

# Duality of codes supported on regular lattices, with an application to enumerative combinatorics

Alberto Ravagnani\*

Institut de Mathématiques  
Université de Neuchâtel  
Emile-Argand 11, CH-2000 Neuchâtel, Switzerland

## Abstract

We construct a family of weight functions on finite abelian groups that yield invertible MacWilliams identities for additive codes. The weights are obtained composing a suitable support map with the rank function of a graded lattice that satisfies certain regularity properties. We express the Krawtchouk coefficients of the corresponding MacWilliams transformation in terms of the combinatorial invariants of the underlying lattice, and show that the most relevant weight functions studied in coding theory belong, up to equivalence, to the class that we introduce. In particular, we compute some classical Krawtchouk coefficients employing a simple combinatorial method. Our approach also allows to systematically construct weight functions that endow the underlying group with a metric space structure. We establish a Singleton-like bound for additive codes, and call optimal the codes that attain the bound. Then we prove that the dual of an optimal code is optimal, and that the weight distribution of an optimal code is completely determined by three fundamental parameters. Finally, we apply MacWilliams identities for the rank weight to enumerative combinatorics problems, computing the number of matrices of given rank over a finite field that satisfy certain linear conditions.

## Introduction

In coding theory, a MacWilliams identity expresses a linear transformation between the weight distribution of a code and the weight distribution of the dual code. MacWilliams identities are named after Jessie MacWilliams, who first discovered relations of this type for linear codes with the Hamming weight (see [22]). Analogous identities were later established for several classes of codes by many authors.

Following e.g. [30], [3] and [10], an additive code  $\mathcal{C} \subseteq G$  is a subgroup of a finite abelian group  $G$ , and the dual code  $\mathcal{C}^* \subseteq \hat{G}$  is its character-theoretic annihilator. Code and dual code are subsets of different ambient spaces, and therefore their weight distributions refer in general to different weight functions, say  $\omega$  and  $\tau$ , on  $G$  and  $\hat{G}$ , respectively. Invertible MacWilliams identities hold when  $\omega$  and  $\tau$  are mutually compatible. In this case the  $\omega$ -distribution of a code  $\mathcal{C}$  and the  $\tau$ -distribution of the dual code  $\mathcal{C}^*$  determine each other. The linear relations between the weight distributions are expressed by certain complex numbers called Krawtchouk coefficients. Their existence is guaranteed

---

\*E-mail: [alberto.ravagnani@unine.ch](mailto:alberto.ravagnani@unine.ch). The author was partially supported by the Swiss National Science Foundation through grant no. 200021\_150207.

by the compatibility of the weight functions, but providing an explicit formula for them is difficult in general.

From the discussion above it apperas that two main problems in the area of MacWilliams identities for additive codes over groups are the following:

1. construct families of weight functions that are mutually compatible,
2. explicitly compute the associated Krawtchouk coefficients.

In the language of group partitions (see [10] and the references within), the two problems read as follows: construct families of Fourier-reflexive partitions on groups, and explicitely compute the associated Krawtchouk matrices. Several weight functions that are classically studied in coding theory provide examples of mutually compatible weights, and the various Krawtchouk coefficients have been computed by different authors employing *ad hoc* techniques.

Employing lattice theory, in this paper we introduce a class of mutually compatible weight functions on finite abelian groups, and study it employing combinatorial techniques.

A regular support is a function defined over a finite abelian group  $G$  that takes values in a graded lattice  $\mathcal{L}$  with certain regularity properties. A regular support induces a weight on  $G$  via the rank function of  $\mathcal{L}$ . We show that a regular support  $\sigma$  on  $G$  with values in  $\mathcal{L}$  induces a regular support  $\sigma^*$  on the character group  $\hat{G}$  with values in the dual lattice  $\mathcal{L}^*$ . This yields in particular a weight function on  $\hat{G}$  via the rank function of  $\mathcal{L}^*$ . In this framework we prove that the weight functions on  $G$  and  $\hat{G}$  induced by  $\sigma$  and  $\sigma^*$ , respectively, are mutually compatible. Moreover, we express the corresponding Krawtchouk coefficients in terms of certain combinatorial invariants of the lattice  $\mathcal{L}$ . In many interesting examples such combinatorial invariants are very easy to compute.

The most relevant weight functions studied in coding theory (such as the Hamming weight, the rank weight, the Lee weight on  $\mathbb{Z}_4$ , the exact weight on  $\mathbb{F}_2^n$  and the homogeneous weight on certain Frobenius rings) belong, up to equivalence, to the family of weights that we introduce. This will allow us to compute the associated Krawtchouk coefficients with a simple combinatorial technique.

Using a specific example of support function (which we call *chain support*) we show how one can systematically construct Fourier-reflexive partitions and weight functions that endow the underlying finite abelian group with a metric space structure. This is particularly interesting for applications in coding theory.

We then consider groups  $G$  equipped with the weight function  $\omega$  induced by a regular support, and establish a Singleton-like bound for additive code  $\mathcal{C} \subseteq G$ . A code is optimal if it attains the bound. We prove that the dual of an optimal code is optimal, and show that the  $\omega$ -distribution of an optimal code is determined by three fundamental parameters.

Finally, we study applications of MacWilliams identities for the rank weight to enumerative problems of matrices. We provide closed formulas for the number of  $k \times m$  matrices over  $\mathbb{F}_q$  with given rank and satisfying one of the following conditions: a prescribed set of their diagonal entries are zero, a prescribed set of their entries sum to zero, their entries are zero in a rectangular region, they are symmetric, they are skew-symmetric. This generalizes some known results obtained with different methods by other authors.

The structure of the paper is as follows. In Section 1 we introduce codes, weight functions and partitions of groups. In Section 2 we briefly recall some results on finite posets, and introduce regular lattices. We define and study regular supports in Section 3. In Section 4 we show that regular supports produce compatible pairs of weights, and express the corresponding Krawtchouk coefficients in terms of certain combinatorial invariants of a lattice. In Section 5 we apply our approach to several weight functions that are studied in coding theory. Optimal codes are studied in Section 6, and enumerative problems of matrices are discussed in Section 7.

# 1 Groups, codes, and compatible weights

Let  $(G, +)$  be a group. The **character group** of  $G$ , denoted by  $(\hat{G}, \cdot)$ , is the set of group homomorphisms  $\chi : G \rightarrow \mathbb{C}^*$  endowed with pointwise multiplication, i.e., for  $\chi_1, \chi_2 \in \hat{G}$ ,

$$(\chi_1 \cdot \chi_2)(g) := \chi_1(g)\chi_2(g), \quad \text{for all } g \in G.$$

The neutral element of  $(\hat{G}, \cdot)$  is the **trivial character**  $\varepsilon \equiv 1$  of  $G$ . The groups  $G$  and  $\hat{G}$  are canonically isomorphic via the map  $\psi : G \rightarrow \hat{G}$  defined, for  $g \in G$ , by  $\psi(g)(\chi) := \chi(g)$  for all  $\chi \in \hat{G}$ . It is well-known that when  $(G, +)$  is finite and abelian the groups  $(G, +)$  and  $(\hat{G}, \cdot)$  are isomorphic, not canonically in general. In particular,  $|G| = |\hat{G}|$ . Notice that for all  $n \geq 1$  we have  $\widehat{G^n} = \hat{G}^n$ , where  $(\chi_1, \dots, \chi_n) \in \widehat{G^n}$  is defined, for all  $(g_1, \dots, g_n) \in G^n$ , by

$$(\chi_1, \dots, \chi_n)(g_1, \dots, g_n) := \prod_{i=1}^n \chi_i(g_i).$$

**Definition 1.** Let  $G$  be a finite abelian group. A **code** in  $G$  is a subgroup  $\mathcal{C} \subseteq G$ . The **dual** of  $\mathcal{C}$  is the code  $\mathcal{C}^* := \{\chi \in \hat{G} : \chi(g) = 1 \text{ for all } g \in \mathcal{C}\} \subseteq \hat{G}$ . We say that  $\mathcal{C}$  is **trivial** if  $\mathcal{C} = \{0\}$  or  $\mathcal{C} = G$ . The code **generated** by codes  $\mathcal{C}, \mathcal{D} \subseteq G$  is the code  $\mathcal{C} + \mathcal{D} := \{c + d : c \in \mathcal{C}, d \in \mathcal{D}\} \subseteq G$ .

The following remark summarizes some properties of duality. The proof is left to the reader.

**Remark 2.** Let  $\mathcal{C} \subseteq G$  be a code. Then  $|\mathcal{C}| \cdot |\mathcal{C}^*| = |G| = |\hat{G}|$ . Moreover, identifying  $G$  and  $\hat{G}$  we have  $\mathcal{C}^{**} = \mathcal{C}$ . Finally, duality and sum of codes relate as follows.

1. Let  $\mathcal{C}, \mathcal{D} \subseteq G$  be codes. Then  $|\mathcal{C} + \mathcal{D}| = |\mathcal{C}| \cdot |\mathcal{D}| / |\mathcal{C} \cap \mathcal{D}|$ .
2. Let  $\mathcal{C}_1, \dots, \mathcal{C}_t \subseteq G$  be codes,  $t \geq 2$ . We have  $\bigcap_{i=1}^t \mathcal{C}_i^* = (\sum_{i=1}^t \mathcal{C}_i)^*$ .

**Definition 3.** Let  $G$  be a finite abelian group. A **weight** on  $G$  is a function  $\omega : G \rightarrow X$ , where  $X$  is a finite set. The  $\omega$ -**distribution** of a code  $\mathcal{C} \subseteq G$  is the collection  $\{W_a(\mathcal{C}, \omega) : a \in X\}$ , where  $W_a(\mathcal{C}, \omega) := |\{g \in \mathcal{C} : \omega(g) = a\}|$  for all  $a \in X$ .

Let  $\omega : G \rightarrow X$  and  $\tau : \hat{G} \rightarrow Y$  be weights. We say that  $(\omega, \tau)$  is a **compatible pair** if for every  $g \in G$  and  $b \in \tau(\hat{G})$  the complex number

$$\sum_{\substack{\chi \in \hat{G} \\ \tau(\chi) = b}} \chi(g)$$

only depends on  $\omega(g)$ . If this is the case, then the **Krawtchouk coefficients** associated to  $(\omega, \tau)$  are defined, for every  $a \in \omega(G)$  and  $b \in \tau(\hat{G})$ , by

$$K(\omega, \tau)(a, b) := \sum_{\substack{\chi \in \hat{G} \\ \tau(\chi) = b}} \chi(g),$$

where  $g \in G$  is any element with  $\omega(g) = a$ . When  $a \notin \omega(G)$  or  $b \notin \tau(\hat{G})$  we put  $K(\omega, \tau)(a, b) := 0$ .

**Remark 4.** Let  $\omega : G \rightarrow X, \tau : \hat{G} \rightarrow Y$  be weights. Identifying  $G$  and  $\hat{G}$  one has  $g(\chi) = \chi(g)$  for all  $g \in G$  and  $\chi \in \hat{G}$ . Thus when  $(\tau, \omega)$  is a compatible pair the Krawtchouk coefficients associated to  $(\tau, \omega)$  are defined, for every  $a \in \tau(\hat{G})$  and  $b \in \omega(G)$ , by

$$K(\tau, \omega)(a, b) = \sum_{\substack{g \in G \\ \omega(g) = b}} \chi(g),$$

where  $\chi \in \hat{G}$  is any character with  $\tau(\chi) = a$ . Again, if  $a \notin \tau(\hat{G})$  or  $b \notin \omega(G)$  then  $K(\tau, \omega)(a, b) = 0$ .

**Definition 5.** Let  $\omega : G \rightarrow X$  be a weight. For all  $a \in \omega(G)$  define  $P_a := \{g \in G : \omega(g) = a\}$ . Then

$$\mathcal{P}(\omega) := \bigsqcup_{a \in \omega(G)} P_a$$

is the **partition** of  $G$  induced by  $\omega$ . We say that weight functions  $\omega : G \rightarrow X$  and  $\omega' : G \rightarrow X'$  are **equivalent** if  $\mathcal{P}(\omega) = \mathcal{P}(\omega')$ , and in this case we write  $\omega \sim \omega'$ .

**Remark 6.** Let  $\omega : G \rightarrow X$ ,  $\omega' : G \rightarrow X'$ ,  $\tau : \hat{G} \rightarrow Y$  and  $\tau' : \hat{G} \rightarrow Y'$  be weights with  $\omega \sim \omega'$  and  $\tau \sim \tau'$ . There exist permutations  $\pi : \omega'(G) \rightarrow \omega(G)$  and  $\eta : \tau'(\hat{G}) \rightarrow \tau(\hat{G})$  such that  $\omega = \pi \circ \omega'$  and  $\tau = \eta \circ \tau'$ . Moreover, it is easy to see that if  $(\omega, \tau)$  is a compatible pair, then  $(\omega', \tau')$  is also a compatible pair, and for all  $a \in \omega'(G)$  and  $b \in \tau'(\hat{G})$  one has

$$K(\omega', \tau')(a, b) = K(\omega, \tau)(\pi(a), \eta(b)).$$

Thus the Krawtchouk coefficients associated to  $(\omega', \tau')$  are essentially the same as the Krawtchouk coefficients associated to  $(\omega, \tau)$ , up to a suitable permutation. Thus some authors prefer to directly concentrate on group partitions when studying Krawtchouk coefficients (see e.g. [10]).

In coding theory however, given a “numerical” weight function  $\omega : G \rightarrow X \subseteq \mathbb{N}$ , one naturally attempts to define a distance  $d_\omega$  on  $G$  by setting  $d_\omega(g, g') := \omega(g - g')$  for all  $g, g' \in G$ . It is easy to construct groups  $G$  and weights  $\omega, \omega' : G \rightarrow X \subseteq \mathbb{N}$  such that  $\omega \sim \omega'$ ,  $d_\omega$  is a distance function, but  $d_{\omega'}$  is not. This is the reason why we prefer to work with weighs. From the point of view of the study of Krawtchouk coefficients, the partition approach and the weight approach are equivalent.

Compatible pairs of weights produce MacWilliams-type identities as follows.

**Theorem 7** (MacWilliams Identities). Let  $G$  be a finite abelian group, and let  $\omega : G \rightarrow X$  and  $\tau : \hat{G} \rightarrow Y$  be weights. Assume that  $(\omega, \tau)$  is compatible. Then for all codes  $\mathcal{C} \subseteq G$  we have

$$W_b(\mathcal{C}^*, \tau) = \frac{1}{|\mathcal{C}|} \sum_{a \in X} K(\omega, \tau)(a, b) W_a(\mathcal{C}, \omega).$$

for all  $b \in Y$ . In particular, the  $\omega$ -distribution of  $\mathcal{C}$  determines the  $\tau$ -distribution of  $\mathcal{C}^*$ .

*Proof.* In the language of [10], we have that  $\mathcal{P}(\omega)$  is finer than  $\widehat{\mathcal{P}(\tau)}$ , the dual of the partition induced by  $\tau$  on  $\hat{G}$  (see [10], Definition 2.1). Thus the result is implied by [10], Theorem 2.7, along with the observation that follows its proof.  $\square$

**Remark 8.** The fact that a pair  $(\omega, \tau)$  is compatible does not imply in general that  $(\tau, \omega)$  is also compatible. This corresponds to the fact that the MacWilliams transformation is not always invertible. The most interesting scenario is when both  $(\omega, \tau)$  and  $(\tau, \omega)$  are compatible, i.e., when  $\omega$  and  $\tau$  are **mutually** compatible. In this case the MacWilliams identities are invertible, and the  $\omega$ -distribution of a code and the  $\tau$ -distribution of the dual code determine each other.

Remark 6 and Remark 8 inspire the following problems in the area of MacWilliams identities over groups. The first two problems have already been mentioned in the introduction of this paper.

(P1) Construct weights  $\omega : G \rightarrow X$  and  $\tau : \hat{G} \rightarrow Y$  such that both  $(\omega, \tau)$  and  $(\tau, \omega)$  are compatible.

(P2) Compute the associated Krawtchouk coefficients.

(P3) Construct weights  $\omega, \tau$  such that both  $(\omega, \tau)$  and  $(\tau, \omega)$  are compatible, and both  $d_\omega$  and  $d_\tau$  are distance functions.

In this paper we introduce a family of weight functions  $\omega, \tau$  such that both  $(\omega, \tau)$  and  $(\tau, \omega)$  are compatible pairs. Moreover, for such weight functions we provide a combinatorial description of the corresponding Krawtchouk coefficients in terms of the invariants of a certain lattice (see Theorem 27). It turns out that the most relevant weight functions studied in coding theory belong to the family that we introduce up to equivalence (see Section 5). We also construct, for any finite abelian group  $G$ , weight functions  $\omega : G \rightarrow X \subseteq \mathbb{N}$  and  $\tau : \hat{G} \rightarrow Y \subseteq \mathbb{N}$  such that  $(\omega, \tau)$  and  $(\tau, \omega)$  are both compatible, and such that  $d_\omega$  and  $d_\tau$  are both distance functions (see Example 32). The main tool that we employ is lattice theory.

We conclude this section mentioning the product weight and the symmetrized weight induced by a weight function. See also Definition 3.1 and 3.2 of [10].

**Definition 9.** Let  $\omega : G \rightarrow X$  be a weight, and let  $n \geq 1$  be an integer.

1. The **product weight** on  $G^n$  associated to  $\omega$  is the function  $\omega^n : G^n \rightarrow X^n$  defined, for all  $(g_1, \dots, g_n)$ , by  $\omega^n(g_1, \dots, g_n) := (\omega(g_1), \dots, \omega(g_n))$ .
2. Choose an enumeration  $X = \{x_0, \dots, x_r\}$  and for all  $(c_1, \dots, c_n) \in X^n$  let  $\text{cmp}(c) := (e_0, \dots, e_r)$ , where  $e_i := |\{1 \leq j \leq n : c_j = i\}|$  for all  $0 \leq i \leq r$ . The **symmetrized weight** on  $G^n$  associated to  $\omega$  is the function  $\omega_{\text{sym}}^n : G^n \rightarrow \{0, \dots, n\}^{r+1}$  defined, for all  $(g_1, \dots, g_n) \in G^n$ , by  $\omega_{\text{sym}}^n(g_1, \dots, g_n) := \text{cmp}(\omega^n(g_1, \dots, g_n))$ .

Compatibility of pairs is preserved by products and symmetrization, as we now show. The first formula of the following proposition also appears in the proof of [10], Theorem 3.3(a).

**Proposition 10.** Let  $\omega : G \rightarrow X$  and  $\tau : \hat{G} \rightarrow Y$  be weights, and let  $n \geq 1$  be an integer. Set  $r := |X|$  and  $s := |Y|$ . Assume that  $(\omega, \tau)$  is a compatible pair. Then  $(\omega^n, \tau^n)$  and  $(\omega_{\text{sym}}^n, \tau_{\text{sym}}^n)$  are compatible pairs. Moreover, for all  $a = (a_1, \dots, a_n) \in X^n$  and  $b = (b_1, \dots, b_n) \in Y^n$ , and for all  $d = (d_0, \dots, d_r) \in \{1, \dots, n\}^{r+1}$  and  $e \in \{1, \dots, n\}^{s+1}$  we have:

$$K(\omega^n, \tau^n)(a, b) = \prod_{j=1}^n K(\omega, \tau)(a_j, b_j),$$

$$K(\omega_{\text{sym}}^n, \tau_{\text{sym}}^n)(d, e) = \sum_{\substack{b \in X^n \\ \text{cmp}(b) = e}} \prod_{j=1}^{d_0} K(\omega, \tau)(0, b_j) \prod_{j=d_0+1}^{d_1} K(\omega, \tau)(1, b_j) \cdots \prod_{j=d_{r-1}+1}^{d_r} K(\omega, \tau)(r, b_j).$$

*Proof.* Let  $(a_1, \dots, a_n) \in \omega^n(G^n)$  and  $(b_1, \dots, b_n) \in \tau^n(\hat{G}^n)$ . For any element  $(g_1, \dots, g_n) \in G^n$  with  $\omega^n(g_1, \dots, g_n) = (a_1, \dots, a_n)$  one has

$$\sum_{\substack{(\chi_1, \dots, \chi_n) \in \hat{G}^n \\ \tau^n(\chi_1, \dots, \chi_n) = (b_1, \dots, b_n)}} (\chi_1, \dots, \chi_n)(g_1, \dots, g_n) = \prod_{j=1}^n K(\omega, \tau)(a_j, b_j). \quad (1)$$

This shows that  $(\omega^n, \tau^n)$  is a compatible pair, and proves the first formula in the statement. Now we study the symmetrized weight. Let  $(d_0, \dots, d_r) \in \omega_{\text{sym}}^n(G^n)$  and  $(e_0, \dots, e_s) \in \tau_{\text{sym}}^n(\hat{G}^n)$ , and let  $(g_1, \dots, g_n) \in G^n$  with  $\omega_{\text{sym}}^n(g_1, \dots, g_n) = (d_0, \dots, d_r)$ . Using (1) we compute

$$\sum_{\substack{(\chi_1, \dots, \chi_n) \in \hat{G}^n \\ \tau_{\text{sym}}^n(\chi_1, \dots, \chi_n) = (e_0, \dots, e_s)}} (\chi_1, \dots, \chi_n)(g_1, \dots, g_n) = \sum_{\substack{(b_1, \dots, b_n) \in X^n \\ \text{cmp}(b_1, \dots, b_n) = (e_0, \dots, e_s)}} \prod_{j=1}^n K(\omega, \tau)(a_j, b_j), \quad (2)$$

where  $(a_1, \dots, a_n) := \omega^n(g_1, \dots, g_n)$ . Up to a permutation of the entries of  $(a_1, \dots, a_n)$ , without loss of generality we may assume  $a_i \leq a_{i+1}$  for all  $1 \leq i \leq n-1$ . Therefore (2) becomes

$$\sum_{\substack{(b_1, \dots, b_n) \in X^n \\ \text{cmp}(b_1, \dots, b_n) = (e_0, \dots, e_s)}} \prod_{j=1}^{d_0} K(\omega, \tau)(0, b_j) \prod_{j=d_0+1}^{d_1} K(\omega, \tau)(1, b_j) \cdots \prod_{j=d_{r-1}+1}^{d_r} K(\omega, \tau)(r, b_j).$$

The expression above only depends on  $(d_0, \dots, d_r)$  and  $(e_0, \dots, e_s)$ . This shows that  $(\omega_{\text{sym}}^n, \tau_{\text{sym}}^n)$  is compatible, and proves the second formula in the statement.  $\square$

Proposition 10 shows that the computation of the Krawtchouk coefficients of the pairs  $(\omega^n, \tau^n)$  and  $(\omega_{\text{sym}}^n, \tau_{\text{sym}}^n)$  reduces to the computation of the Krawtchouk coefficients of  $(\omega, \tau)$ .

## 2 Regular lattices

In this section we briefly recall some basic notions on posets and lattices, and propose a definition of regular lattice. See Chapter 3 of [28] for a general introduction to posets. Throughout this paper we only treat finite lattices.

Given a poset  $(L, \leq)$  and  $S, T \in L$ , we write  $S < T$  for  $S \leq T$  and  $S \neq T$ . We write  $S \triangleleft T$  if  $S < T$  and there is no  $U \in L$  with  $S < U < T$ . In this case we say that  $T$  **covers**  $S$ .

**Definition 11.** A **lattice** is a poset  $(L, \leq)$  where every  $S, T \in L$  have a unique meet and a unique join, denoted by  $S \wedge T$  and  $S \vee T$ , respectively.

Meet and join of a lattice  $\mathcal{L} = (L, \leq, \wedge, \vee)$  define two binary, commutative and associative operations  $\wedge, \vee : L \times L \rightarrow L$ . In particular, for any non-empty finite subset  $M \subseteq L$ , the lattice elements  $\bigwedge\{S : S \in M\}$  and  $\bigvee\{S : S \in M\}$  are well-defined. When  $\mathcal{L}$  is **finite** (i.e.,  $L$  is finite), we set  $0_{\mathcal{L}} := \bigwedge\{S : S \in L\}$  and  $1_{\mathcal{L}} := \bigvee\{S : S \in L\}$ .

A finite lattice  $\mathcal{L}$  is **graded** of **rank**  $r$  if all maximal chains (with respect to inclusion) in  $\mathcal{L}$  have length  $r$ . We denote the rank of a graded lattice  $\mathcal{L}$  by  $\text{rk}(\mathcal{L})$ .

**Remark 12.** Let  $\mathcal{L} = (L, \leq, \wedge, \vee)$  be a finite graded lattice of rank  $r$ . There exists a unique function  $\rho_{\mathcal{L}} : L \rightarrow \{0, \dots, r\}$ , called the **rank function** of  $\mathcal{L}$ , with  $\rho_{\mathcal{L}}(0_{\mathcal{L}}) = 0$  and  $\rho_{\mathcal{L}}(T) = \rho_{\mathcal{L}}(S) + 1$  whenever  $S \triangleleft T$  ([28], page 281). The function  $\rho_{\mathcal{L}}$  is monotonic, i.e.,  $\rho_{\mathcal{L}}(S) \leq \rho_{\mathcal{L}}(T)$  whenever  $S \leq T$ . Moreover,  $\rho_{\mathcal{L}}(L) = \{0, \dots, r\}$ , and  $0_{\mathcal{L}}$  and  $1_{\mathcal{L}}$  are the only elements of rank 0 and  $r$ , respectively.

The **dual** of a lattice  $\mathcal{L} = (L, \leq, \wedge, \vee)$  is the lattice  $\mathcal{L}^* = (L, \preceq, \wedge, \vee)$ , where  $S \preceq T$  if and only if  $T \leq S$ ,  $\wedge := \vee$  and  $\vee := \wedge$ . If  $\mathcal{L}$  is finite (and so  $\mathcal{L}^*$  is finite) then  $0_{\mathcal{L}^*} = 1_{\mathcal{L}}$  and  $1_{\mathcal{L}^*} = 0_{\mathcal{L}}$ . Clearly,  $\mathcal{L}^{**} = \mathcal{L}$ . Notice moreover that  $\mathcal{L}$  is graded if and only if  $\mathcal{L}^*$  is graded. If this is the case, then  $\text{rk}(\mathcal{L}) = \text{rk}(\mathcal{L}^*)$  and  $\rho_{\mathcal{L}^*}(S) = \text{rk}(\mathcal{L}) - \rho_{\mathcal{L}}(S)$  for all  $S \in L$ .

**Definition 13.** Let  $\mathcal{L} = (L, \leq)$  be a finite poset. The **Möbius function** of  $\mathcal{L}$  is the function  $\mu_{\mathcal{L}} : \{(S, T) \in L \times L : S \leq T\} \rightarrow \mathbb{Z}$  inductively defined by  $\mu_{\mathcal{L}}(S, S) = 1$  for all  $S \in L$ , and

$$\mu_{\mathcal{L}}(S, T) = - \sum_{S \leq U < T} \mu_{\mathcal{L}}(S, U) \quad \text{for all } S, T \in L \text{ with } S < T.$$

Using the fact that a lattice  $\mathcal{L}$  and its dual lattice  $\mathcal{L}^*$  are anti-isomorphic, one can show that  $\mu_{\mathcal{L}^*}(S, T) = \mu_{\mathcal{L}}(T, S)$  for all  $S, T \in \mathcal{L}$  (see e.g. [27], Proposition 2.1.10).

Now we introduce regular lattices.

**Definition 14.** A finite graded lattice  $\mathcal{L} = (L, \leq, \wedge, \vee)$  of rank  $r$  is **regular** if the following hold.

- (a) For all  $T \in L$  and for all integers  $0 \leq s \leq r$ ,
  - the number of  $S \in L$  with  $\rho_{\mathcal{L}}(S) = s$  and  $S \leq T$  only depends on  $s$  and  $\rho_{\mathcal{L}}(T)$ ,
  - the number of  $S \in L$  with  $\rho_{\mathcal{L}}(S) = s$  and  $T \leq S$  only depends on  $s$  and  $\rho_{\mathcal{L}}(T)$ .
- (b) For all  $S, T \in L$  with  $S \leq T$ , the Möbius function  $\mu_{\mathcal{L}}(S, T)$  only depends on  $\rho_{\mathcal{L}}(S)$  and  $\rho_{\mathcal{L}}(T)$ .

The main combinatorial invariants of a regular lattice are defined as follows.

**Notation 15.** Let  $\mathcal{L} = (L, \leq, \wedge, \vee)$  be a regular lattice of rank  $r$ . Then for all integers  $0 \leq s, t \leq r$  we set

$$\mu_{\leq}(s, t) := |\{S \in L : S \leq T, \rho_{\mathcal{L}}(S) = s\}| \quad \text{and} \quad \mu_{\geq}(s, t) := |\{S \in L : T \leq S, \rho_{\mathcal{L}}(S) = s\}|,$$

where  $T \in L$  is any element with  $\rho_{\mathcal{L}}(T) = t$ . For given integers  $0 \leq s \leq t \leq r$  we also define

$$\mu_{\mathcal{L}}(s, t) := \mu_{\mathcal{L}}(S, T),$$

where  $S, T \in L$  are any lattice elements with  $S \leq T$ ,  $\rho_{\mathcal{L}}(S) = s$ , and  $\rho_{\mathcal{L}}(T) = t$ . For all  $s > t$  we set  $\mu_{\mathcal{L}}(s, t) := 0$ .

A different notion of lattice regularity was proposed by Delsarte in [8]. The definition of Delsarte is motivated by coding theory applications via association schemes. Our approach is different from the approach of [8].

The following result easily follows from the definitions and from the properties of the Möbius function. It expresses the parameters of the dual of a regular lattice  $\mathcal{L}$  in terms of the parameters of  $\mathcal{L}$ .

**Proposition 16.** Let  $\mathcal{L} = (L, \leq, \wedge, \vee)$  be a regular lattice of rank  $r$ . Then  $\mathcal{L}^* = (L, \preceq, \wedge, \vee)$  is regular of rank  $r$ , and for all  $0 \leq s, t \leq r$  we have

$$\mu_{\preceq}(s, t) = \mu_{\geq}(r - s, r - t), \quad \mu_{\succeq}(s, t) = \mu_{\leq}(r - s, r - t), \quad \text{and} \quad \mu_{\mathcal{L}^*}(s, t) = \mu_{\mathcal{L}}(r - t, r - s).$$

We conclude the section with a convenient criterion to test regularity of a lattice without employing the Möbius function.

**Proposition 17.** Let  $\mathcal{L} = (L, \leq, \wedge, \vee)$  be a finite graded lattice. Assume that for every  $S, T \in L$  with  $S \leq T$  and for every  $\rho_{\mathcal{L}}(S) \leq i \leq \rho_{\mathcal{L}}(T)$  the number  $\{U \in L : S \leq U \leq T \text{ and } \rho_{\mathcal{L}}(U) = i\}$  only depends on  $i$ ,  $\rho_{\mathcal{L}}(S)$  and  $\rho_{\mathcal{L}}(T)$ . Then  $\mathcal{L}$  is regular.

*Proof.* Property (a) of Definition 14 is immediate, and property (b) can be proved by induction on  $\rho_{\mathcal{L}}(T) - \rho_{\mathcal{L}}(S)$  using the definition of the Möbius function.  $\square$

### 3 Regular supports and duality

In this section we propose a definition of regular support on a finite abelian group, and establish some properties that we need in the sequel. In particular, we show that a regular support on a finite abelian group  $G$  induces a regular support on the character group  $\hat{G}$ .

**Notation 18.** If  $G$  is a group,  $\mathcal{L} = (L, \leq)$  is a poset and  $\sigma : G \rightarrow L$  is any function, then for all  $S \in L$  we set  $G_{\sigma}(S) := \{g \in G : \sigma(g) \leq S\}$ .

**Definition 19.** Let  $(G, +)$  be a finite abelian group, and let  $\mathcal{L} = (L, \leq, \wedge, \vee)$  be a regular lattice. A **regular support** on  $G$  with values in  $\mathcal{L}$  is a function  $\sigma : G \rightarrow L$  that satisfies the following.

- (A)  $\sigma(g) = 0_{\mathcal{L}}$  if and only if  $g = 0$ .
- (B)  $\sigma(g) = \sigma(-g)$  for all  $g \in G$ .
- (C)  $\sigma(g_1 + g_2) \leq \sigma(g_1) \vee \sigma(g_2)$  for all  $g_1, g_2 \in G$ .
- (D)  $G_{\sigma}(S_1 \vee S_2) = G_{\sigma}(S_1) + G_{\sigma}(S_2)$  for all  $S_1, S_2 \in L$ .
- (E) For all  $S \in L$ ,  $|G_{\sigma}(S)|$  only depends on  $\rho_{\mathcal{L}}(S)$ .

**Notation 20.** We denote a regular lattice on  $G$  with values in  $\mathcal{L}$  by  $\sigma : G \dashrightarrow \mathcal{L}$ . Moreover, for all  $0 \leq s \leq r$  we set

$$\gamma_{\sigma}(S) := |G_{\sigma}(S)|,$$

where  $S \in L$  is any element with  $\rho_{\mathcal{L}}(S) = s$ . Given a lattice element  $S \in L$  and a code  $\mathcal{C} \subseteq G$ , we define  $\mathcal{C}_{\sigma}(S) := G_{\sigma}(S) \cap \mathcal{C}$ .

Now we show that the definition of regular lattice behaves well under dualization.

**Notation 21.** Let  $\sigma : (G, +) \dashrightarrow \mathcal{L} = (L, \leq, \wedge, \vee)$  be a regular support. Define the function  $\sigma^* : \hat{G} \rightarrow L$  by

$$\sigma^*(\chi) := \bigvee \{S \in L : \chi \in G_{\sigma}(S)^*\}$$

for all  $\chi \in \hat{G}$ . Since  $G_{\sigma}(0_{\mathcal{L}}) = \{0\}$  by property (A) of Definition 19, we have  $\chi \in G_{\sigma}(0_{\mathcal{L}})^*$  for any  $\chi \in \hat{G}$ . This shows that  $\sigma^*(\chi)$  is well-defined. We regard  $\sigma^*$  as a function on  $\hat{G}$  with values in  $\mathcal{L}^*$ . In particular, according to Notation 18, for  $S \in L$  we have

$$\hat{G}_{\sigma^*}(S) = \{\chi \in \hat{G} : \sigma^*(\chi) \preceq S\}.$$

**Lemma 22.** Let  $\sigma : (G, +) \dashrightarrow \mathcal{L} = (L, \leq, \wedge, \vee)$  be a regular support. Then for all  $\chi \in \hat{G}$  we have  $\chi \in G_{\sigma}(\sigma^*(\chi))^*$ . Equivalently,  $\sigma^*(\chi)$  is the maximum  $S \in L$  such that  $\chi \in G_{\sigma}(S)^*$ .

*Proof.* Let  $\chi \in \hat{G}$  be any character. As already shown,  $\{S \in L : \chi \in G_{\sigma}(S)^*\} \neq \emptyset$ . Choose an enumeration  $\{S \in L : \chi \in G_{\sigma}(S)^*\} = \{S_1, S_2, \dots, S_t\}$ . By property (D) of Definition 19 and the associativity of the join we have  $G_{\sigma}(S_1 \vee S_2 \vee \dots \vee S_t) = G_{\sigma}(S_1) + G_{\sigma}(S_2) + \dots + G_{\sigma}(S_t)$ . Thus Remark 2 implies  $G_{\sigma}(S_1 \vee S_2 \vee \dots \vee S_t)^* = G_{\sigma}(S_1)^* \cap G_{\sigma}(S_2)^* \cap \dots \cap G_{\sigma}(S_t)^*$ . Since  $\chi \in G_{\sigma}(S_i)^*$  for all  $i \in \{1, \dots, t\}$ , we have  $\chi \in G_{\sigma}(\sigma^*(\chi))^*$ , as claimed.  $\square$

The following crucial theorem summarizes the main properties of a regular support. In particular, it shows that a support on a group  $G$  with values in a lattice  $\mathcal{L}$  induces a regular support on the character group  $\hat{G}$  with values in the dual lattice  $\mathcal{L}^*$ .

**Theorem 23.** Let  $\sigma : (G, +) \dashrightarrow \mathcal{L} = (L, \leq, \wedge, \vee)$  be regular. The following hold.

1.  $G_{\sigma}(S)^* = \hat{G}_{\sigma^*}(S)$  for all  $S \in L$ .
2. The map  $\chi \mapsto \sigma^*(\chi)$  defines a regular support  $\sigma^* : (\hat{G}, \cdot) \dashrightarrow \mathcal{L}^* = (L, \preceq, \wedge, \vee)$ .
3.  $\gamma_{\sigma^*}(s) = |G|/\gamma_{\sigma}(\text{rk}(\mathcal{L}) - s)$  for all  $0 \leq s \leq \text{rk}(\mathcal{L})$ .
4. Identifying  $\hat{\hat{G}}$  and  $G$  we have  $\sigma^{**} = \sigma$ .



**Definition 24.** The regular support  $\sigma^* : (\hat{G}, \cdot) \dashrightarrow \mathcal{L}^*$  defined by part 2 of Theorem 23 and Notation 21 is called the **dual support** of  $\sigma$ .

*Proof of Theorem 23.* 1. Take any  $S \in L$ . If  $\chi \in G_\sigma(S)^*$  then, by definition,  $S \leq \sigma^*(\chi)$ , i.e.,  $\sigma^*(\chi) \preceq S$ . This shows  $G_\sigma(S)^* \subseteq \hat{G}_{\sigma^*}(S)$ . Now assume that  $\chi \in \hat{G}_{\sigma^*}(S)$ , and let  $g \in G_\sigma(S)$ . We have  $\sigma(g) \leq S \leq \sigma^*(\chi)$ , and so  $g \in G_\sigma(\sigma^*(\chi))$ . Lemma 22 implies  $\chi(g) = 1$ , and so  $\hat{G}_{\sigma^*}(S) \subseteq G_\sigma(S)^*$ .

2. The lattice  $\mathcal{L}^*$  is regular by Proposition 16, and the group  $(\hat{G}, \cdot)$  is finite and abelian. Let  $\varepsilon$  be the trivial character of  $G$ . By 1 we have  $\hat{G}_{\sigma^*}(0_{\mathcal{L}^*}) = G_\sigma(1_{\mathcal{L}})^* = G^* = \{\varepsilon\}$ , and this proves property (A) of Definition 19. For  $\chi \in \hat{G}$  and  $S \in L$  we have  $\chi \in G_\sigma(S)^*$  if and only if  $1/\chi \in G_\sigma(S)^*$ . By definition of dual support, this gives property (B). Now take any  $\chi_1, \chi_2 \in \hat{G}$ , and let  $g \in G_\sigma(\sigma^*(\chi_1)) \cap G_\sigma(\sigma^*(\chi_2))$ . Lemma 22 implies  $\chi_1(g) = \chi_2(g) = 1$ , and so  $(\chi_1 \cdot \chi_2)(g) = \chi_1(g)\chi_2(g) = 1$ . Thus

$$\chi_1 \cdot \chi_2 \in (G_\sigma(\sigma^*(\chi_1)) \cap G_\sigma(\sigma^*(\chi_2)))^* = G_\sigma(\sigma^*(\chi_1) \wedge \sigma^*(\chi_2))^*,$$

where the last equality directly follows from the definition of meet. As a consequence we have  $\sigma^*(\chi_2) \wedge \sigma^*(\chi_1) \leq \sigma^*(\chi_1 \cdot \chi_2)$ , i.e.,  $\sigma^*(\chi_1 \cdot \chi_2) \preceq \sigma^*(\chi_1) \vee \sigma^*(\chi_2)$ . This establishes property (C). Let  $S_1, S_2 \in L$ . By definition of meet we have  $G_\sigma(S_1 \wedge S_2) = G_\sigma(S_1) \cap G_\sigma(S_2)$ . Taking the duals, by Remark 2 we obtain  $G_\sigma(S_1 \wedge S_2)^* = G_\sigma(S_1)^* \cdot G_\sigma(S_2)^*$ , and part 1 of the statement gives  $\hat{G}_{\sigma^*}(S_1 \wedge S_2) = \hat{G}_{\sigma^*}(S_1) \cdot \hat{G}_{\sigma^*}(S_2)$ , i.e.,  $\hat{G}_{\sigma^*}(S_1 \vee S_2) = \hat{G}_{\sigma^*}(S_1) \cdot \hat{G}_{\sigma^*}(S_2)$ . This is property (D). Let  $S \in L$ . By part 1 and Remark 2 we have  $|\hat{G}_{\sigma^*}(S)| = |G|/|G_\sigma(S)|$ . Thus  $|\hat{G}_{\sigma^*}(S)|$  only depends on  $\rho_{\mathcal{L}^*}(S) = \text{rk}(\mathcal{L}) - \rho_{\mathcal{L}}(S)$ . This is property (E).

3. Let  $r := \text{rk}(\mathcal{L}) = \text{rk}(\mathcal{L}^*)$ . Take  $S \in L$  with  $\rho_{\mathcal{L}^*}(S) = s$ . Part 1 and Remark 2 imply  $\hat{G}_{\sigma^*}(S)^* = G_\sigma(S)$ . Thus  $\gamma_{\sigma^*}(s) = |\hat{G}_{\sigma^*}(S)| = |G|/|\hat{G}_{\sigma^*}(S)^*| = |G|/|G_\sigma(S)| = |G|/\gamma_\sigma(s)$ .
4. As before, part 1 and Remark 2 give  $\hat{G}_{\sigma^*}(S)^* = G_\sigma(S)$  for all  $S \in L$ . Hence, for all  $g \in G$ ,

$$\sigma^{**}(g) = \bigvee \{S \in L : g \in \hat{G}_{\sigma^*}(S)^*\} = \bigwedge \{S \in L : g \in G_\sigma(S)\} = \bigwedge \{S \in L : \sigma(g) \leq S\} = \sigma(g).$$

This concludes the proof.  $\square$

We close this section with an example that shows that every finite abelian group admits regular supports.

**Example 25** (Chain support). Let  $(L, \leq)$  be a finite chain, and let  $S_0 < S_1 < \dots < S_r$  be the elements of  $L$ . For all  $i, j \in \{0, \dots, r\}$  define  $S_i \wedge S_j := S_{\min\{i, j\}}$  and  $S_i \vee S_j := S_{\max\{i, j\}}$ . Then  $\mathcal{L} = (L, \leq, \wedge, \vee)$  is regular lattice of rank  $r$  with:

$$\mu_{\leq}(s, t) = \begin{cases} 1 & \text{if } s \leq t \\ 0 & \text{else} \end{cases} \quad \mu_{\geq}(s, t) = \begin{cases} 1 & \text{if } s \geq t \\ 0 & \text{else} \end{cases} \quad \mu_{\mathcal{L}}(s, t) = \begin{cases} 1 & \text{if } s = t \\ -1 & \text{if } t = s + 1 \\ 0 & \text{else} \end{cases}$$

for all  $0 \leq s, t \leq r$ . Now let  $(G, +)$  be a finite abelian group, and let  $\mathcal{L} = (L, \subseteq, \wedge, \vee)$  be a chain of subgroups of  $G$ , i.e.,  $\{0\} = G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_r = G$ , endowed with the structure of regular lattice described above. The **chain support**  $\sigma : G \dashrightarrow \mathcal{L}$  is the function  $\sigma : G \rightarrow L$  defined, for all  $g \in G$ , by  $\sigma(g) := G_i$ , where  $i = \min\{0 \leq j \leq r : g \in G_j\}$ . It is easy to check that  $\sigma$  is a regular support. By definition,  $G_\sigma(G_s) = G_s$  for all  $0 \leq s \leq r$ , and therefore  $\gamma_\sigma(s) = |G_s|$  for all  $s$ . Moreover, for any  $\chi \in \hat{G}$  we have  $\sigma^*(\chi) = G_i$ , where  $i = \max\{0 \leq j \leq r : \chi \in G_j^*\}$ .

## 4 Compatible weights from regular supports

A regular support  $\sigma : G \dashrightarrow \mathcal{L}$  induces a weight function on  $G$  via the rank function of the regular lattice  $\mathcal{L}$ .

**Definition 26.** Let  $\sigma : (G, +) \dashrightarrow \mathcal{L}$  be a regular support. The  $\sigma$ -**weight** on  $\mathcal{C}$  induced by  $\sigma$  is the function  $\omega_\sigma : G \rightarrow \{0, \dots, \text{rk}(\mathcal{L})\}$  defined by  $\omega_\sigma(g) := \rho_{\mathcal{L}}(\sigma(g))$  for all  $g \in G$ .

Now we state our main result.

**Theorem 27.** Let  $\sigma : (G, +) \dashrightarrow \mathcal{L}$  be a regular support,  $r = \text{rk}(\mathcal{L})$ . The following hold.

1. The pair  $(\omega_{\sigma^*}, \omega_\sigma)$  is compatible. Moreover, for all  $i \in \omega_{\sigma^*}(\hat{G})$  and  $j \in \omega_\sigma(G)$  we have

$$K(\omega_{\sigma^*}, \omega_\sigma)(i, j) = \sum_{s=0}^r \gamma_\sigma(s) \mu_{\mathcal{L}}(s, j) \mu_{\leq}(s, r-i) \mu_{\geq}(j, s).$$

2. The pair  $(\omega_\sigma, \omega_{\sigma^*})$  is compatible. Moreover, for all  $i \in \omega_\sigma(G)$  and  $j \in \omega_{\sigma^*}(\hat{G})$  we have

$$K(\omega_\sigma, \omega_{\sigma^*})(i, j) = |G| \sum_{s=0}^r \frac{1}{\gamma_\sigma(r-s)} \mu_{\mathcal{L}}(r-j, r-s) \mu_{\geq}(r-s, i) \mu_{\leq}(r-j, r-s).$$

**Remark 28.** Theorem 27 shows that a regular support  $\sigma : G \dashrightarrow \mathcal{L}$  automatically yields compatible pairs of weights  $(\omega_\sigma, \omega_{\sigma^*})$  and  $(\omega_{\sigma^*}, \omega_\sigma)$  on  $G$  and  $\hat{G}$ . Moreover, it expresses the associated Krawtchouk coefficients in terms of the combinatorial invariants of the lattice  $\mathcal{L}$ . This provides an answer to problems (P1) and (P2) on page 4. As we will see, in many relevant examples such combinatorial invariants are very easy to determine. In those cases Theorem 27 gives an effective method to compute the Krawtchouk coefficients.

*Proof of Theorem 27.* Throughout this proof, a sum over an empty set of indices is zero by definition. Let us first show part 1. Part 2 will follow easily. Fix any character  $\chi \in \hat{G}$ , and let  $f, g : L \rightarrow \mathbb{C}$  be the complex-valued functions defined by

$$f(T) := \sum_{\substack{g \in G \\ \sigma(g)=T}} \chi(g), \quad g(T) := \sum_{S \leq T} f(S) \quad \text{for all } T \in L.$$

By the orthogonality relations of characters (see e.g. [21], Lemma 1.1.32), for all  $T \in L$  we have

$$g(T) = \sum_{S \leq T} f(S) = \sum_{g \in G_\sigma(T)} \chi(g) = \begin{cases} \gamma_\sigma(\rho_{\mathcal{L}}(T)) & \text{if } \chi \in G_\sigma(T)^* \\ 0 & \text{if } \chi \notin G_\sigma(T)^*. \end{cases}$$

Therefore applying the Möbius inversion formula ([28], Proposition 3.7.1) to  $f$  and  $g$  we obtain

$$\begin{aligned} f(T) &= \sum_{\substack{S \leq T \\ \chi \in G_\sigma(S)^*}} \gamma_\sigma(\rho_{\mathcal{L}}(S)) \mu_{\mathcal{L}}(S, T) = \sum_{s=0}^r \sum_{\substack{S \leq T \\ \rho_{\mathcal{L}}(\bar{S})=s \\ \chi \in G_\sigma(S)^*}} \gamma_\sigma(s) \mu_{\mathcal{L}}(S, T) \\ &= \sum_{s=0}^r \sum_{\substack{S \leq T \\ \rho_{\mathcal{L}}(\bar{S})=s \\ \chi \in \hat{G}_{\sigma^*}(S)}} \gamma_\sigma(s) \mu_{\mathcal{L}}(S, T), \end{aligned}$$

where the last equality follows from part 1 of Theorem 23. Thus for any integer  $0 \leq j \leq r$  one has

$$\begin{aligned} \sum_{\substack{g \in G \\ \omega_\sigma(g)=j}} \chi(g) &= \sum_{\substack{T \in L \\ \rho_{\mathcal{L}}(T)=j}} f(T) = \sum_{\substack{T \in L \\ \rho_{\mathcal{L}}(T)=j}} \sum_{s=0}^r \sum_{\substack{S \leq T \\ \rho_{\mathcal{L}}(\bar{S})=s \\ \chi \in \hat{G}_{\sigma^*}(S)}} \gamma_\sigma(s) \mu_{\mathcal{L}}(S, T) \\ &= \sum_{s=0}^r \gamma_\sigma(s) \sum_{\substack{T \in L \\ \rho_{\mathcal{L}}(T)=j}} \sum_{\substack{S \leq T \\ \rho_{\mathcal{L}}(\bar{S})=s \\ \chi \in \hat{G}_{\sigma^*}(S)}} \mu_{\mathcal{L}}(S, T). \end{aligned}$$

By the regularity of  $\mathcal{L}$ ,  $\mu_{\mathcal{L}}(S, T) = \mu_{\mathcal{L}}(s, j)$  for all  $S, T \in L$  with  $S \leq T$ ,  $\rho_{\mathcal{L}}(S) = s$  and  $\rho_{\mathcal{L}}(T) = j$ . Thus setting  $\alpha(s, j, \chi) := |\{(S, T) \in L \times L : \rho_{\mathcal{L}}(S) = s, \rho_{\mathcal{L}}(T) = j, S \leq T, \sigma^*(\chi) \preceq S\}|$  we have

$$\sum_{\substack{g \in G \\ \omega_\sigma(g)=j}} \chi(g) = \sum_{s=0}^r \gamma_\sigma(s) \mu_{\mathcal{L}}(s, j) \alpha(s, j, \chi). \quad (3)$$

Now we derive a convenient expression for  $\alpha(s, j, \chi)$ . By definition,

$$\alpha(s, j, \chi) = \sum_{\substack{S \in L \\ \rho_{\mathcal{L}}(S)=s \\ \sigma^*(\chi) \preceq S}} |\{T \in L : \rho_{\mathcal{L}}(T) = j, S \leq T\}| = \sum_{\substack{S \in L \\ \rho_{\mathcal{L}}(S)=s \\ S \leq \sigma^*(\chi)}} \mu_{\geq}(j, s) = \mu_{\leq}(s, \rho_{\mathcal{L}}(\sigma^*(\chi))) \mu_{\geq}(j, s).$$

By the properties of the rank function of the dual lattice (see Section 2) and the definition of  $\omega_{\sigma^*}$  we have  $\rho_{\mathcal{L}}(\sigma^*(\chi)) = r - \rho_{\mathcal{L}^*}(\sigma^*(\chi)) = r - \omega_{\sigma^*}(\chi)$ . It follows  $\mu_{\leq}(s, \rho_{\mathcal{L}}(\sigma^*(\chi))) = \mu_{\leq}(s, r - \omega_{\sigma^*}(\chi))$ , and therefore  $\alpha(s, j, \chi) = \mu_{\leq}(s, r - \omega_{\sigma^*}(\chi)) \mu_{\geq}(j, s)$ . Substituting this expression for  $\alpha(s, j, \chi)$  into equation (3) yields

$$\sum_{\substack{g \in G \\ \omega_\sigma(g)=j}} \chi(g) = \sum_{s=0}^r \gamma_\sigma(s) \mu_{\mathcal{L}}(s, j) \mu_{\leq}(s, r - \omega_{\sigma^*}(\chi)) \mu_{\geq}(j, s).$$

By Remark 4, this shows part 1.

By Theorem 23,  $\sigma^*$  is a regular support, and  $\sigma^{**} = \sigma$  when identifying  $G$  and  $\hat{\hat{G}}$ . Thus part 2 follows from part 1 applied to  $\sigma^* : \hat{G} \dashrightarrow \mathcal{L}^*$ , along with Proposition 16.  $\square$

**Remark 29.** Let  $\sigma : G \dashrightarrow \mathcal{L}$  be a regular support. In the language of [10] one can show that the partitions  $\mathcal{P}(\omega_\sigma)$  and  $\mathcal{P}(\omega_{\sigma^*})$  are both Fourier-reflexive and mutually dual. We do not go into the details of the proof.

Combining Example 25, Theorem 27 and Remark 29 we obtain the following simple result.

**Corollary 30** (Fourier-reflexive partitions via subgroups). Let  $(G, +)$  be a finite abelian group, and let  $\{0\} = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_r = G$  be a chain of subgroups of  $G$ . Then

$$\{0\} \sqcup \bigsqcup_{i=1}^r G_i \setminus G_{i-1}$$

is a Fourier-reflexive partition of  $G$  of cardinality  $r + 1$ .

Under certain assumptions on the lattice  $\mathcal{L}$ , the weight function  $\omega_\sigma$  associated to a regular support  $\sigma : G \dashrightarrow \mathcal{L}$  induces a distance  $d_{\omega_\sigma}$  on  $G$ . Recall that a finite lattice  $\mathcal{L} = (L, \leq, \wedge, \vee)$  is **modular** if for all  $S, T, U \in L$  with  $S \leq U$  one has  $S \vee (T \wedge U) = (S \vee T) \wedge U$ . Clearly, the dual of a modular lattice is modular.

**Proposition 31.** Let  $\sigma : (G, +) \dashrightarrow \mathcal{L}$  be a regular support. If  $\mathcal{L}$  is modular, then the function  $d_{\omega_\sigma} : G \times G \rightarrow \mathbb{N}$  defined by  $d_{\omega_\sigma}(g, g') := \omega_\sigma(g - g')$  for all  $g, g' \in G$  is a distance function.

*Proof.* Write  $d := d_{\omega_\sigma}$ . Let  $g, g' \in G$ . By definition,  $d(g, g') = 0$  if and only if  $\rho_{\mathcal{L}}(\sigma(g - g')) = 0$ . By the properties of  $\rho_{\mathcal{L}}$  (Remark 12), this happens if and only if  $\sigma(g - g') = 0$ , i.e., by property (A) of Definition 19, if and only if  $g = g'$ . By property (B) of Definition 19 we have  $d(g, g') = \omega_\sigma(g - g') = \rho_{\mathcal{L}}(\sigma(g - g')) = \rho_{\mathcal{L}}(\sigma(g' - g)) = \omega_\sigma(g' - g) = d(g', g)$ . Now let  $h, g, g' \in G$ . The rank function of a modular lattice  $\mathcal{L} = (L, \leq, \wedge, \vee)$  satisfies  $\rho_{\mathcal{L}}(S \vee T) = \rho_{\mathcal{L}}(S) + \rho_{\mathcal{L}}(T) - \rho_{\mathcal{L}}(S \wedge T)$  for all  $S, T \in L$  (see [28], page 287). Thus by property (C) of Definition 19 we have

$$d(g, g') = \omega_\sigma(g - g') = \omega_\sigma(g - h - (g' - h)) \leq \rho_{\mathcal{L}}(\sigma(g - h) \vee \sigma(g' - h)) \leq d(g, h) + d(h, g').$$

This concludes the proof.  $\square$

**Example 32** (Chain support, continued). Let  $(G, +)$  be a finite abelian group, and let  $\mathcal{L}$  be a chain  $\{0\} = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_r = G$  of subgroups of  $G$  endowed with the lattice structure described in Example 25. Then  $\mathcal{L}$  is modular. Denote by  $\sigma : G \dashrightarrow \mathcal{L}$  the associated chain support. By Example 25,  $\sigma$  is regular. Thus, by Proposition 31,  $d_{\omega_\sigma}$  is a distance on  $G$ . By Theorem 23,  $\sigma^*$  is a regular support. Since  $\mathcal{L}$  is modular,  $\mathcal{L}^*$  is modular. Hence by Proposition 31  $d_{\omega_{\sigma^*}}$  is a distance on  $\hat{G}$ . By Theorem 27,  $(\omega_\sigma, \omega_{\sigma^*})$  and  $(\omega_{\sigma^*}, \omega_\sigma)$  are compatible pairs such that both  $d_{\omega_\sigma}$  and  $d_{\omega_{\sigma^*}}$  are distance functions. This provides an answer to problem (P3) on page 4.

We conclude the example giving a more explicit description of  $\omega_{\sigma^*}$ . Let  $\nu$  be the chain support on the character group  $\hat{G}$  associated to the chain  $\{1\} = G_r^* \subsetneq G_{r-1}^* \subsetneq \cdots \subsetneq \hat{G}$ , which we denote by  $\mathcal{U}$ . We have  $\omega_\nu = \omega_{\sigma^*}$ . Indeed, as already mentioned in Example 25, for a fixed  $\chi \in \hat{G}$  we have  $\sigma^*(\chi) = G_i$ , where  $i = \max\{0 \leq j \leq r : \chi \in G_j^*\}$ . Thus, by definition,  $\omega_{\sigma^*}(\chi) = \rho_{\mathcal{L}^*}(\sigma^*(\chi)) = r - i$ . On the other hand,

$$\omega_\nu(\chi) = \min\{0 \leq j \leq r : \chi \in G_{r-j}^*\} = r - \max\{0 \leq j \leq r : \chi \in G_j^*\} = r - i = \omega_{\sigma^*}(\chi),$$

as claimed.

## 5 MacWilliams identities in coding theory

In this section we show that many weight functions traditionally studied in coding theory are induced by suitable regular supports up to equivalence. We also employ Theorem 27 to easily compute the corresponding Krawtchouk coefficients with a combinatorial method. Most of them have been computed by other authors employing *ad hoc* techniques in the past. Theorem 27 provides a general method that applies to different contexts. The case of the rank weight (Example 36) is particularly interesting, as the standard method to compute the associated Krawtchouk coefficients is quite sophisticated (see [9]). Theorem 27 allows to compute them in a simple way.

**Example 33** (Additive codes with the Hamming weight). Let  $n \geq 1$  be a positive integer, and let  $[n] := \{1, \dots, n\}$ . Then  $\mathcal{L} = (2^{[n]}, \subseteq, \cap, \cup)$  is a regular lattice of rank  $n$ . The rank function of  $\mathcal{L}$  is the cardinality of sets. The parameters of  $\mathcal{L}$  are given by

$$\mu_{\subseteq}(s, t) = \binom{t}{s}, \quad \mu_{\supseteq}(s, t) = \binom{n-t}{s-t}, \quad \mu_{\mathcal{L}}(s, t) = \begin{cases} (-1)^{t-s} & \text{if } s \leq t \\ 0 & \text{if } s > t \end{cases}$$

for all  $0 \leq s, t \leq n$ . The formula for  $\mu_{\mathcal{L}}(s, t)$  can be easily proved by induction on  $t - s$  with the aid of the Binomial Theorem ([28], page 24). See [28], Example 3.8.3 for a different proof using the product of chains. Let  $(G, +)$  be a finite abelian group. Define the **Hamming support**  $\sigma_H : G^n \rightarrow 2^{[n]}$  by  $\sigma_H(g_1, \dots, g_n) := \{i \in [n] : g_i \neq 0\}$  for all  $(g_1, \dots, g_n) \in G^n$ . The weight induced on  $G^n$  by the Hamming support is the **Hamming weight**  $\omega_H$ . For  $S \subseteq [n]$  and  $(\chi_1, \dots, \chi_n) \in \hat{G}^n$  we have  $(\chi_1, \dots, \chi_n) \in G_\sigma^n(S)^*$  if and only if  $\chi_s$  is the trivial character of  $G$  for all  $s \in S$ . Therefore  $\sigma_H^*(\chi_1, \dots, \chi_n) = \{i \in [n] : \chi_i \text{ is trivial}\}$ . It follows

$$\omega_{\sigma_H^*}(\chi_1, \dots, \chi_n) = n - |\{i \in [n] : \chi_i \text{ is trivial}\}| = |\{i \in [n] : \chi_i \text{ is not trivial}\}|.$$

Thus in the following we write  $\omega_{\sigma_H^*} = \omega_H$ .

Let  $G \subseteq G^n$  be a code. Then Theorem 27 allows to compute the Krawtchouk coefficients for the Hamming weight as

$$K(\omega_H, \omega_H)(i, j) = \sum_{s=0}^n (-1)^{j-s} |G|^s \binom{n-i}{s} \binom{n-s}{j-s}$$

for all  $0 \leq i, j \leq n$ . By Theorem 7, for every code  $\mathcal{C} \subseteq G^n$  and for all  $0 \leq j \leq n$  we have

$$W_j(\mathcal{C}^*, \omega_H) = \frac{1}{|\mathcal{C}|} \sum_{i=0}^n W_i(\mathcal{C}, \omega_H) \sum_{s=0}^n (-1)^{j-s} |G|^s \binom{n-i}{s} \binom{n-s}{j-s}.$$

These are the *MacWilliams identities for the Hamming weight over a group*.

**Example 34** (Linear codes with the Hamming weight). Take  $G = \mathbb{F}_q$  in Example 33. Define the **orthogonal** of a linear code  $\mathcal{C} \subseteq \mathbb{F}_q^n$  by  $\mathcal{C}^\perp := \{v \in \mathbb{F}_q^n : \langle w, v \rangle = 0 \text{ for all } w \in \mathcal{C}\}$ , where  $\langle \cdot, \cdot \rangle$  is the standard inner product of  $\mathbb{F}_q^n$ . One can show that  $W_j(\mathcal{C}^\perp, \omega_H) = W_j(\mathcal{C}^*, \omega_H)$  for all linear codes  $\mathcal{C} \subseteq \mathbb{F}_q^n$ . By Example 33, for all  $0 \leq j \leq n$  we have

$$W_j(\mathcal{C}^\perp, \omega_H) = \frac{1}{|\mathcal{C}|} \sum_{i=0}^n W_i(\mathcal{C}, \omega_H) \sum_{s=0}^n (-1)^{j-s} |G|^s \binom{n-i}{s} \binom{n-s}{j-s}.$$

These are the *MacWilliams identities for linear codes with the Hamming weight*. See for instance Chapter 5 of [24] or Chapter 7 of [16] for equivalent formulations.

**Example 35** (Modified exact weight). Let  $(G, +)$  be a non-trivial finite abelian group. Let  $\sigma$  denote the chain support on  $G$  associated to the chain  $\{0\} \subsetneq G$ . See Example 25. Let  $\omega_\sigma : G \rightarrow \{0, 1\}$  be the induced weight. By the second part of Example 32,  $\omega_{\sigma^*}$  is the weight on  $\hat{G}$  induced by the chain support associated to the chain  $\{1\} \subsetneq \hat{G}$ . If  $n \geq 2$  and  $G = \mathbb{F}_2$ , then the product weight  $\omega_\sigma^n$  is the **exact weight** on  $\mathbb{F}_2^n$  (see [24], page 147). For a general  $G$  we obtain a weight that partitions the elements of the group  $G^n$  according to the positions of their non-zero entries. With the aid of Theorem 27 and Example 25 one easily computes the Krawtchouk coefficients for  $(\omega_\sigma, \omega_{\sigma^*})$  and  $(\omega_{\sigma^*}, \omega_\sigma)$  as

$$K(\omega_\sigma, \omega_{\sigma^*})(i, j) = K(\omega_{\sigma^*}, \omega_\sigma)(i, j) = \begin{cases} 1 & \text{if } j = 0 \\ -1 & \text{if } j = 1 \text{ and } i = 1 \\ |G| - 1 & \text{if } j = 1 \text{ and } i = 0 \end{cases}$$

for all  $i, j \in \{0, 1\}$ . Proposition 10 also allows to compute the coefficients for the product and the symmetrized weight.

**Example 36** (Linear codes with the rank weight). Let  $1 \leq k \leq m$  be integers, and let  $G := \text{Mat}$  be the vector space of  $k \times m$  matrices over  $\mathbb{F}_q$ . Denote by  $L$  the set of all subspaces of  $\mathbb{F}_q^k$ . Then  $\mathcal{L} = (L, \subseteq, \cap, +)$  is a regular lattice of rank  $k$ . Notice that the join is the sum of subspaces. The rank function of  $\mathcal{L}$  is given by  $\rho_{\mathcal{L}}(V) = \dim(V)$  for all  $V \subseteq \mathbb{F}_q^k$  (see [28], page 281). The parameters of  $\mathcal{L}$  are, for all  $0 \leq s, t \leq k$ ,

$$\mu_{\subseteq}(s, t) = \begin{bmatrix} t \\ s \end{bmatrix}, \quad \mu_{\supseteq}(s, t) = \begin{bmatrix} k-t \\ s-t \end{bmatrix}, \quad \mu_{\mathcal{L}}(s, t) = \begin{cases} (-1)^{t-s} q^{\binom{t-s}{2}} & \text{if } s \leq t \\ 0 & \text{if } s > t, \end{cases}$$

where the symbols in squared brackets are the  $q$ -ary binomial coefficients (see e.g. [1]). The formula for  $\mu_{\mathcal{L}}(s, t)$  can be easily proved by induction on  $t-s$  with the aid of the Gaussian Binomial Theorem ([28], equation (1.87) at page 74). An elegant argument that uses the fact that  $\mathcal{L}$  is a geometric lattice can be found in [28], Example 3.10.2. Denote by  $\text{colsp}(M) \subseteq \mathbb{F}_q^k$  the space generated by the columns of a matrix  $M \in \text{Mat}$ . Then  $\sigma_{\text{rk}} : M \mapsto \text{colsp}(M)$  is a regular support  $\sigma_{\text{rk}} : \text{Mat} \dashrightarrow \mathcal{L}$  with  $\gamma_{\sigma}(s) = q^{ms}$  for all  $0 \leq s \leq k$  (see [25], Lemma 26). It is called the **rank support**. Let  $\omega_{\text{rk}} := \omega_{\sigma_{\text{rk}}}$  be the **rank weight**, and set  $\omega_{\text{rk}}^* := \omega_{\sigma_{\text{rk}}^*}$  for ease of notation. Employing Theorem 27, the Krawtchouk coefficients for  $(\omega_{\text{rk}}, \omega_{\text{rk}}^*)$  and  $(\omega_{\text{rk}}^*, \omega_{\text{rk}})$  can be easily computed as

$$K(\omega_{\text{rk}}, \omega_{\text{rk}}^*)(i, j) = K(\omega_{\text{rk}}^*, \omega_{\text{rk}})(i, j) = \sum_{s=0}^k (-1)^{j-s} q^{ms + \binom{j-s}{2}} \begin{bmatrix} k-s \\ k-j \end{bmatrix} \begin{bmatrix} k-i \\ s \end{bmatrix} \quad (4)$$

for all  $0 \leq i, j \leq k$ .

Recall that the **trace-product** of matrices  $M, N \in \text{Mat}$  is  $\langle M, N \rangle := \text{Tr}(MN^t)$ , where  $\text{Tr}$  is the trace of matrices, and the superscript  $t$  denotes transposition. The **orthogonal** of a code  $\mathcal{C} \subseteq \text{Mat}$  is  $\mathcal{C}^{\perp} := \{M \in \text{Mat} : \langle N, M \rangle = 0 \text{ for all } N \in \mathcal{C}\}$ . One can show that if  $\mathcal{C} \subseteq \text{Mat}$  is a linear code, then  $W_j(\mathcal{C}^{\perp}, \omega_{\text{rk}}) = W_j(\mathcal{C}^*, \omega_{\text{rk}}^*)$  for all  $0 \leq j \leq k$ . Thus combining Theorem 7 and equation (4) we obtain

$$W_j(\mathcal{C}^{\perp}, \omega_{\text{rk}}) = \frac{1}{|\mathcal{C}|} \sum_{i=0}^k W_i(\mathcal{C}, \omega_{\text{rk}}) \sum_{s=0}^k (-1)^{j-s} q^{ms + \binom{j-s}{2}} \begin{bmatrix} k-s \\ k-j \end{bmatrix} \begin{bmatrix} k-i \\ s \end{bmatrix}$$

for all  $0 \leq j \leq k$ . These are the *MacWilliams identities for linear codes with the rank weight*, first established by Delsarte in [9]. Rank-metric codes were recently re-discovered for applications in linear network coding (see e.g. [26]).

**Example 37** (Lee weight on  $\mathbb{Z}_4$ ). The **Lee weight** on  $\mathbb{Z}_4$  is the function  $\omega_{\text{Lee}} : \mathbb{Z}_4 \rightarrow \{0, 1, 2\} \subseteq \mathbb{N}$  defined by  $\omega_{\text{Lee}}(0) := 0$ ,  $\omega_{\text{Lee}}(1) = \omega_{\text{Lee}}(3) := 1$  and  $\omega_{\text{Lee}}(2) := 2$ . See [19] and [14] or Chapter 12 of [16] and the references within. Denote by  $\sigma$  be the chain support on  $\mathbb{Z}_4$  associated to the chain  $\{0\} \subsetneq \mathbb{Z}_2 \subsetneq \mathbb{Z}_4$ . Then  $\omega_{\text{Lee}} \sim \omega_{\sigma}$ . Let  $\zeta \in \mathbb{C}$  be a primitive fourth root of unity. Define the map  $\psi : \mathbb{Z}_4 \rightarrow \hat{\mathbb{Z}}_4$  by  $\psi(a)(b) := \zeta^{ab}$  for all  $a, b \in \mathbb{Z}_4$ . Then  $\psi$  is a group isomorphism, and it is natural to define the **Lee weight** on  $\hat{\mathbb{Z}}_4$  by  $\omega_{\text{Lee}}^* := \omega_{\text{Lee}} \circ \psi^{-1}$ . A direct computation shows  $\omega_{\sigma} = \omega_{\sigma^*} \circ \psi$ , and therefore  $\omega_{\text{Lee}}^* = \omega_{\text{Lee}} \circ \psi^{-1} \sim \omega_{\sigma} \circ \psi^{-1} = \omega_{\sigma^*} \circ \psi \circ \psi^{-1} = \omega_{\sigma^*}$ . Thus the Krawtchouk coefficients associated to  $(\omega_{\text{Lee}}, \omega_{\text{Lee}}^*)$  are the same as the Krawtchouk coefficients associated to  $(\omega_{\sigma}, \omega_{\sigma^*})$ , up to a permutation. They can be explicitly computed combining Example 25 and Theorem 27 as follows. We write  $K_{\text{Lee}}$  for  $K(\omega_{\text{Lee}}, \omega_{\text{Lee}}^*)$ .

$$\begin{array}{lll} K_{\text{Lee}}(0, 0) = 1 & K_{\text{Lee}}(0, 1) = 2 & K_{\text{Lee}}(0, 2) = 1 \\ K_{\text{Lee}}(1, 0) = 1 & K_{\text{Lee}}(1, 1) = 0 & K_{\text{Lee}}(1, 2) = -1 \\ K_{\text{Lee}}(2, 0) = 1 & K_{\text{Lee}}(2, 1) = -2 & K_{\text{Lee}}(2, 2) = 1. \end{array}$$

Proposition 10 also allows to compute the Krawtchouk coefficients for the **symmetrized Lee weight** on the product group  $\mathbb{Z}_4^n$ , for  $n \geq 1$ .

**Example 38** (Homogeneous weight on certain Frobenius rings). We denote the socle and the Jacobson radical of a finite (possibly non-commutative) Frobenius ring  $R$  by  $\text{soc}(R)$  and  $\text{rad}(R)$ , respectively. See Chapter 6 of [18] for the main properties of Frobenius rings, or [12] and [11] for a coding theory approach. It is known that  $\text{rad}(R)$  is a two-sided ideal, and that  $\text{soc}(R) \cong R/\text{rad}(R)$  as rings. Moreover, if  $R$  is local, i.e.,  $\text{rad}(R)$  is the unique maximal left and right ideal of  $R$ , then  $R/\text{rad}(R)$  is a field, called the *residue field*.

Let  $R := R_1 \times R_2 \times \cdots \times R_n$ , where each  $R_i$  is a finite local Frobenius ring with residue field  $R/\text{rad}(R_i) \cong \text{soc}(R_i)$  of order  $q$ . Then  $R$  is Frobenius with  $\text{soc}(R) = \prod_{i=1}^n \text{soc}(R_i)$ . The values of the **homogeneous weight**  $\omega_{\text{hom}} : R \rightarrow \mathbb{R}$  (see [7], [12], and [15]) on  $R$  were explicitly computed in [11], Proposition 3.8, as

$$\omega_{\text{hom}}(a) = \begin{cases} 1 - \left(\frac{-1}{q-1}\right)^{\text{wt}(a)} & \text{if } a \in \text{soc}(R) \\ 1 & \text{otherwise,} \end{cases}$$

where  $\text{wt}(a) := |\{1 \leq i \leq n : a_i \neq 0\}|$  is the weight of  $a = (a_1, \dots, a_n)$ .

From now on we assume  $q \geq 3$ . In particular, we have  $\omega_{\text{hom}}(a) = 0$  if and only if  $a = 0$ . Let  $[n+1] := \{1, \dots, n+1\}$ , and  $L := \{S \subseteq [n+1] : n+1 \notin S\} \cup \{[n+1]\}$ . Then  $\mathcal{L} = (L, \subseteq, \cap, \cup)$  is a regular lattice of rank  $n+1$ , where the rank function is given by the cardinality of sets. It is easy to see that the parameters of  $\mathcal{L}$  are, for all  $0 \leq s, t \leq n+1$ ,

$$\mu_{\subseteq}(s, t) = \begin{cases} \binom{t}{s} & \text{if } s \leq t \leq n \\ \binom{n}{s} & \text{if } s \leq n, t = n+1 \\ 1 & \text{if } s = t = n+1 \\ 0 & \text{if } s > t, \end{cases} \quad \mu_{\supseteq}(s, t) = \begin{cases} \binom{n-t}{s-t} & \text{if } t \leq s \leq n \\ 1 & \text{if } t \leq s = n+1 \\ 0 & \text{if } s < t, \end{cases}$$

$$\mu_{\mathcal{L}}(s, t) = \begin{cases} (-1)^{t-s} & \text{if } s \leq t \leq n \\ 0 & \text{if } t < s, \text{ or } t = n+1 \text{ and } s < n \\ -1 & \text{if } t = n+1, s = n. \end{cases}$$

The formula for  $\mu_{\mathcal{L}}(s, t)$  can be proved by induction on  $t - s$  using the Binomial Theorem, as in Example 33. Define  $\sigma : R \rightarrow L$  by  $\sigma(a) := [n+1]$  if  $a \notin \text{soc}(R)$ , and  $\sigma(a) := \{1 \leq i \leq n : a_i \neq 0\}$  if  $a \in \text{soc}(R)$ . One can check that  $\sigma : R \dashrightarrow \mathcal{L}$  is a regular support with

$$\gamma_{\sigma}(s) = \begin{cases} q^s & \text{if } s \leq n \\ |R| & \text{if } s = n+1 \end{cases}$$

for all  $0 \leq s \leq n+1$ . Moreover,  $\omega_{\sigma} \sim \omega_{\text{hom}}$ . By Definition 5 and Remark 29, in the language of [10] we have

$$\mathcal{P}(\omega_{\text{hom}}) = \mathcal{P}(\omega_{\sigma}), \quad \widehat{\mathcal{P}(\omega_{\text{hom}})} = \mathcal{P}(\omega_{\sigma^*}).$$

Thus the Krawtchouk matrix  $\mathbf{K}$  associated to the homogeneous weight partition (see Section 4 of [11]) is given by

$$\mathbf{K}_{ij} := K(\omega_{\sigma^*}, \omega_{\sigma})(i, j) \tag{5}$$

for all  $0 \leq i, j \leq 2$ . Notice that  $\mathbf{K}$  is defined up to a permutation of rows and columns. With the aid of Theorem 27, for  $n = 1$  one obtains

$$\mathbf{K} = \begin{bmatrix} 1 & q-1 & |R|-q \\ 1 & q-1 & -q \\ 1 & -1 & 0 \end{bmatrix}.$$

The same matrix appears in [11], and in [4] for  $R = \mathbb{Z}_8$ . To the extent of our knowledge the general formula that one obtains combining equation (5) and Theorem 27 is new. Notice moreover that  $\mathcal{L}$  is modular, and therefore  $\omega_\sigma$  automatically induces a distance function on  $R$  by Proposition 31.

For simpler Frobenius rings we can express the homogenous weight via a suitable chain support on the ring. For example, the homogeneous weight on a finite local Frobenius ring  $R$  is equivalent to the chain support associated to the chain  $0 \subsetneq \text{soc}(R) \subsetneq R$  (see [2] or [11] for the values of the homogeneous weight on such rings).

## 6 Optimality

In this section we establish a Singleton-like bound for additive codes endowed with the weight function induced by regular supports. A code is optimal when its parameters attain the bound. We show that the dual of an optimal code is optimal, and that the distribution of an optimal code is determined by its parameters. This extends classical and well-known results for linear codes with the Hamming and the rank weight.

Throughout this section,  $\sigma : (G, +) \dashrightarrow \mathcal{L} = (L, \leq, \wedge, \vee)$  is a regular support, and  $r$  denotes the rank of  $\mathcal{L}$ .

**Definition 39.** Let  $\mathcal{C} \subseteq G$  be a non zero code. The **minimum weight** of  $\mathcal{C} \subseteq G$  is the integer  $d_\omega(\mathcal{C}) := \min\{\omega(g) : g \in \mathcal{C}, g \neq 0\}$ .

We start with the Singleton-like bound.

**Proposition 40.** Let  $\mathcal{C} \subseteq G$  be a non-zero code of minimum weight  $d$ . We have  $|\mathcal{C}| \leq |G|/\gamma_\sigma(d-1)$ .

*Proof.* Take any  $S \in L$  with  $\rho_\mathcal{L}(S) = d - 1$ . We have  $\mathcal{C} \cap G_\sigma(S) = \{0\}$ . Remark 2 implies  $|\mathcal{C} + G_\sigma(S)| = |\mathcal{C}| \cdot |G_\sigma(S)| = |\mathcal{C}| \cdot \gamma_\sigma(d-1)$ . Clearly,  $\mathcal{C} + G_\sigma(S) \subseteq G$ , and so  $|\mathcal{C} + G_\sigma(S)| \leq |G|$ . The bound follows.  $\square$

A non-zero code  $\mathcal{C} \subseteq G$  is **optimal** if it attains the bound of Proposition 40. We need the following preliminary result.

**Lemma 41.** Let  $\mathcal{C} \subseteq G$  be a code. Take any  $S \in L$ , and let  $s := \rho_\mathcal{L}(S)$ . Then

$$|\mathcal{C}_\sigma(S)| = \frac{|\mathcal{C}| \cdot |\mathcal{C}_{\sigma^*}^*(S)|}{\gamma_{\sigma^*}(r-s)}.$$

*Proof.* By definition,  $\mathcal{C}_\sigma(S) = G_\sigma(S) \cap \mathcal{C}$ , and Remark 2 implies

$$|\mathcal{C}_\sigma(S)| = \frac{|G_\sigma(S)| \cdot |\mathcal{C}|}{|G_\sigma(S) + \mathcal{C}|} = \frac{|G_\sigma(S)| \cdot |\mathcal{C}| \cdot |(G_\sigma(S) + \mathcal{C})^*|}{|G|}. \quad (6)$$

Again by Remark 2 we have  $|(G_\sigma(S) + \mathcal{C})^*| = |G_\sigma(S)^* \cap \mathcal{C}^*| = |\hat{G}_{\sigma^*}(S) \cap \mathcal{C}^*|$ , where the last equality follows from Theorem 23. Since  $\hat{G}_{\sigma^*}(S) \cap \mathcal{C}^* = \mathcal{C}_{\sigma^*}^*(S)$  by definition, equation (6) can be written as

$$|\mathcal{C}_\sigma(S)| = \frac{|G_\sigma(S)| \cdot |\mathcal{C}| \cdot |\mathcal{C}_{\sigma^*}^*(S)|}{|G|}.$$

The result now follows from  $|G|/|G_\sigma(S)| = |G|/\gamma_\sigma(s) = \gamma_{\sigma^*}(r-s)$ , again by Theorem 23.  $\square$

**Theorem 42.** Let  $\mathcal{C} \subseteq G$  be a non-trivial optimal code. Then  $d_{\omega_{\sigma^*}}(\mathcal{C}^*) \geq r - d_{\omega_\sigma}(\mathcal{C}) + 2$ , and the code  $\mathcal{C}^*$  is optimal.



*Proof.* Let  $d := d_{\omega_\sigma}(\mathcal{C})$  and  $d^* := d_{\omega_{\sigma^*}}(\mathcal{C}^*)$ . Since  $\mathcal{C}$  is an optimal code, we have  $|\mathcal{C}| = |G|/\gamma_\sigma(d-1)$ . Remark 2 and Theorem 23 imply

$$|\mathcal{C}^*| = \gamma_\sigma(d-1) = |\hat{G}|/\gamma_{\sigma^*}(r-d+1). \quad (7)$$

Let  $S \in L$  be any element with  $\rho_{\mathcal{L}^*}(S) = r-d+1$ . Then  $\rho_{\mathcal{L}}(S) = r - (r-d+1) = d-1$ , and so  $\mathcal{C}_\sigma(S) = \{0\}$ . Lemma 41 gives

$$|\mathcal{C}_{\sigma^*}^*(S)| = \frac{|\mathcal{C}_\sigma(S)| \cdot \gamma_{\sigma^*}(r-d+1)}{|\mathcal{C}|} = 1,$$

where the last equality easily follows from equation (7) and Remark 2. Thus  $\mathcal{C}_{\sigma^*}^*(S) = \{0\}$ , and so the minimum weight of  $\mathcal{C}^*$  satisfies  $d^* \geq r-d+2$ . In particular,  $\gamma_{\sigma^*}(d^*-1) \geq \gamma_{\sigma^*}(r-d+1)$ . Therefore combining Proposition 40 applied to  $\mathcal{C}^*$  and  $\sigma^*$  with equation (7) we obtain

$$\frac{|\hat{G}|}{\gamma_{\sigma^*}(r-d+1)} \geq \frac{|\hat{G}|}{\gamma_{\sigma^*}(d^*-1)} \geq |\mathcal{C}^*| = \frac{|\hat{G}|}{\gamma_{\sigma^*}(r-d+1)}.$$

It follows  $|\mathcal{C}^*| = |\hat{G}|/\gamma_{\sigma^*}(d^*-1)$ , i.e.,  $\mathcal{C}^*$  is optimal.  $\square$

We conclude this section showing that the distribution of an optimal code is completely determined by its parameters. Our proof uses MacWilliams identities written in a form that is less explicit than Theorem 7, but more convenient in this case. We start with a preliminary lemma.

**Lemma 43.** Let  $\mathcal{C} \subseteq G$  be a code. Then for all integers  $0 \leq s \leq r$  we have

$$\sum_{\substack{S \in L \\ \rho_{\mathcal{L}}(S)=s}} |\mathcal{C}_\sigma(S)| = \sum_{i=0}^r W_i(\mathcal{C}, \omega_\sigma) \mu_{\geq}(s, i).$$

*Proof.* We will count the elements of the set  $\mathcal{A} := \{(c, S) \in \mathcal{C} \times L : \sigma(c) \leq S, \rho_{\mathcal{L}}(S) = s\}$  in two different ways. On the one hand,

$$|\mathcal{A}| = \sum_{\substack{S \in L \\ \rho_{\mathcal{L}}(S)=s}} |\{g \in \mathcal{C} : \sigma(g) \leq S\}| = \sum_{\substack{S \in L \\ \rho_{\mathcal{L}}(S)=s}} |\mathcal{C}_\sigma(S)|.$$

On the other hand,

$$|\mathcal{A}| = \sum_{g \in \mathcal{C}} |\{S \in L : \sigma(g) \leq S, \rho_{\mathcal{L}}(S) = s\}| = \sum_{i=0}^r \sum_{\substack{g \in \mathcal{C} \\ \omega_\sigma(g)=i}} \mu_{\geq}(s, i) = \sum_{i=0}^r W_i(\mathcal{C}, \omega_\sigma) \mu_{\geq}(s, i),$$

and the formula follows.  $\square$

**Theorem 44** (Implicit MacWilliams Identities). Let  $\mathcal{C} \subseteq G$  be a code. For all integers  $0 \leq s \leq r$  we have

$$\sum_{i=0}^s W_i(\mathcal{C}, \omega_\sigma) \mu_{\geq}(s, i) = \frac{|\mathcal{C}|}{\gamma_{\sigma^*}(r-s)} \sum_{j=0}^{r-s} W_j(\mathcal{C}^*, \omega_{\sigma^*}) \mu_{\leq}(s, r-j).$$

*Proof.* Fix an integer  $0 \leq s \leq r$ . We have

$$\sum_{i=0}^r W_i(\mathcal{C}, \omega_\sigma) \mu_{\geq}(s, i) = \sum_{\substack{S \in L \\ \rho_{\mathcal{L}}(S)=s}} |\mathcal{C}_\sigma(S)| = \frac{|\mathcal{C}|}{\gamma_{\sigma^*}(r-s)} \sum_{\substack{S \in L \\ \rho_{\mathcal{L}}(S)=s}} |\mathcal{C}_{\sigma^*}^*(S)|, \quad (8)$$

where the two equalities follow from Lemma 43 and 41, respectively. For any  $S \in L$  we have  $\rho_{\mathcal{L}}(S) = s$  if and only if  $\rho_{\mathcal{L}^*}(S) = r - s$ . Moreover, since  $\sigma^*$  is a regular support by Theorem 23, Lemma 43 applied to  $\mathcal{C}^*$  and  $\sigma^*$  gives

$$\sum_{\substack{S \in L \\ \rho_{\mathcal{L}}(S)=s}} |\mathcal{C}_{\sigma^*}^*(S)| = \sum_{\substack{S \in L \\ \rho_{\mathcal{L}^*}(S)=r-s}} |\mathcal{C}_{\sigma^*}^*(S)| = \sum_{j=0}^r W_j(\mathcal{C}^*, \omega_{\sigma^*}) \mu_{\geq}(r-s, j) = \sum_{j=0}^r W_j(\mathcal{C}^*, \omega_{\sigma^*}) \mu_{\leq}(s, r-j), \quad (9)$$

where the last equality follows from Proposition 16. Combining (8) and (9) we obtain

$$\sum_{i=0}^r W_i(\mathcal{C}, \omega_\sigma) \mu_{\geq}(s, i) = \frac{|\mathcal{C}|}{\gamma_{\sigma^*}(r-s)} \sum_{j=0}^r W_j(\mathcal{C}^*, \omega_{\sigma^*}) \mu_{\leq}(s, r-j).$$

For  $i > s$  and  $j > r - s$  we have  $\mu_{\geq}(s, i) = \mu_{\leq}(s, r-j) = 0$ , and the theorem follows.  $\square$

Now we can show that the  $\omega_\sigma$ -distribution of an optimal code is determined by its parameters.

**Corollary 45.** Let  $\mathcal{C} \subseteq G$  a non-trivial optimal code. The  $\omega_\sigma$ -distribution of  $\mathcal{C}$  is determined by  $|G|$ ,  $r$  and  $d_{\omega_\sigma}(\mathcal{C})$ .

*Proof.* Define  $d := d_{\omega_\sigma}(\mathcal{C})$  and  $d^* := d_{\omega_{\sigma^*}}(\mathcal{C}^*)$ . Since  $\mathcal{C}$  is optimal, we have  $d^* \geq r - d + 2$  by Theorem 42. As a consequence, for all integers  $d \leq s \leq r$  we have  $0 \leq r - s \leq r - d \leq d^* - 2 < d$ . In particular, Theorem 44 implies

$$\mu_{\geq}(s, 0) + \sum_{i=d}^s W_i(\mathcal{C}, \omega_\sigma) \mu_{\geq}(s, i) = \frac{|G|}{\gamma_\sigma(d-1) \gamma_{\sigma^*}(r-s)} \mu_{\leq}(s, r)$$

for all  $d \leq s \leq r$ . This corresponds to a linear system of  $r - d + 1$  equations in the  $r - d + 1$  unknowns  $W_d(\mathcal{C}, \omega_\sigma), W_{d+1}(\mathcal{C}, \omega_\sigma), \dots, W_r(\mathcal{C}, \omega_\sigma)$ , and one can see that the matrix of the system is lower triangular with all ones on the diagonal.  $\square$

## 7 Enumerative problems of matrices

In this section we show how one can employ the MacWilliams identities for the rank weight to answer some enumerative combinatorics questions on matrices over a finite field.

Following the notation of Example 36, in the sequel  $k$  and  $m$  are integers with  $1 \leq k \leq m$ , and  $\mathbb{F}_q$  is the finite field with  $q$  elements. We denote by  $\text{Mat}$  the  $km$ -dimensional vector space of  $k \times m$  matrices over  $\mathbb{F}_q$ . Given an integer  $s \geq 0$  we set  $[s] := \{1, \dots, s\}$ . The rank support on  $\text{Mat}$  is denoted by  $\sigma_{\text{rk}}$ , and  $\omega_{\text{rk}}$  is the rank weight. We write “rank-distribution” for “ $\omega_{\text{rk}}$ -distribution”. Up to a transposition,  $k \leq m$  is not restrictive. All dimensions are computed over  $\mathbb{F}_q$ .

The **trace-product** of matrices  $M, N \in \text{Mat}$  is defined by  $\langle M, N \rangle := \text{Tr}(MN^t)$ , where  $\text{Tr}$  denotes the trace of matrices and  $t$  denotes transposition. The **orthogonal** of a linear code  $\mathcal{C} \subseteq \text{Mat}$  is the linear code  $\mathcal{C}^\perp = \{M \in \text{Mat} : \langle M, N \rangle = 0 \text{ for all } N \in \mathcal{C}\}$ . Let us first recall the MacWilliams identities (see [9] or Example 36).

**Theorem 46.** Let  $\mathcal{C} \subseteq \text{Mat}$  be a linear code. The rank-distributions of  $\mathcal{C}$  and  $\mathcal{C}^\perp$  satisfy

$$W_j(\mathcal{C}^\perp, \omega_{\text{rk}}) = \frac{1}{|\mathcal{C}|} \sum_{i=0}^k W_i(\mathcal{C}, \omega_{\text{rk}}) \sum_{s=0}^k (-1)^{j-s} q^{ms + \binom{j-s}{2}} \begin{bmatrix} k-s \\ k-j \end{bmatrix} \begin{bmatrix} k-i \\ s \end{bmatrix}$$

for all  $0 \leq j \leq k$ . In particular, they determine each other.

The first enumerative technique that we employ is very simple. If  $f : \text{Mat} \rightarrow \mathbb{F}_q$  is a non-zero  $\mathbb{F}_q$ -linear function, then  $\ker(f)^\perp$  is a linear code generated by one matrix. Any two generating matrices have the same rank, say  $R_f$ . Thus the rank distribution of the linear code  $\mathcal{C} := \ker(f)^\perp$  is

$$W_i(\mathcal{C}, \omega_{\text{rk}}) = \begin{cases} 1 & \text{if } i = 0 \\ q-1 & \text{if } i = R_f \\ 0 & \text{otherwise.} \end{cases}$$

Therefore applying Theorem 46 to  $\mathcal{C} := \ker(f)^\perp$  one can explicitly compute the number of matrices of rank  $j$  in  $\ker(f) = \mathcal{C}^\perp$  for all  $0 \leq j \leq k$ . More precisely, the following hold.

**Corollary 47.** Let  $f : \text{Mat} \rightarrow \mathbb{F}_q$  be a non-zero linear map, and let  $R_f$  be the rank of any matrix that generates  $\ker(f)^\perp$ . For all  $0 \leq j \leq k$  the number of rank  $j$  matrices in  $\ker(f)$  is

$$\frac{1}{q} \sum_{s=0}^k (-1)^{j-s} q^{ms + \binom{j-s}{2}} \begin{bmatrix} k-s \\ k-j \end{bmatrix} \left( \begin{bmatrix} k \\ s \end{bmatrix} + (q-1) \begin{bmatrix} k-R_f \\ s \end{bmatrix} \right).$$

**Example 48.** Let  $f : \text{Mat} \rightarrow \mathbb{F}_q$  be the linear map that sends a matrix to the sum of its entries. The orthogonal of  $\ker(f)$  is generated by the matrix whose entries are all ones. The rank of such matrix is clearly one. Therefore for all  $0 \leq j \leq k$  the number of rank  $j$  matrices over  $\mathbb{F}_q$  of size  $k \times m$  whose entries sum to zero is

$$\frac{1}{q} \sum_{s=0}^k (-1)^{j-s} q^{ms + \binom{j-s}{2}} \begin{bmatrix} k-s \\ k-j \end{bmatrix} \left( \begin{bmatrix} k \\ s \end{bmatrix} + (q-1) \begin{bmatrix} k-1 \\ s \end{bmatrix} \right).$$

It is now easy to extend Example 48 and obtain the following result.

**Corollary 49.** Let  $I \subseteq [k] \times [m]$  be a non-zero set of indices. For all  $0 \leq j \leq k$  the number of  $k \times m$  rank  $j$  matrices  $M$  over  $\mathbb{F}_q$  such that  $\sum_{(s,t) \in I} M_{st} = 0$  is

$$\frac{1}{q} \sum_{s=0}^k (-1)^{j-s} q^{ms + \binom{j-s}{2}} \begin{bmatrix} k-s \\ k-j \end{bmatrix} \left( \begin{bmatrix} k \\ s \end{bmatrix} + (q-1) \begin{bmatrix} k - \text{rk}(M(I)) \\ s \end{bmatrix} \right),$$

where  $M(I)$  denotes the  $k \times m$  matrix defined, for all  $(s, t) \in [k] \times [m]$ , by  $M(I)_{st} := 1$  if  $(s, t) \in I$ , and  $M(I)_{st} := 0$  otherwise.

The computation of the number of matrices over  $\mathbb{F}_q$  with given size, rank and zero entries in a prescribed region is an active research area in combinatorics and combinatorial statistics (see e.g. [13], [17], [20], [29] and the references within). Such matrices can be regarded as  $q$ -analogues of permutations with restricted positions. It turns out that some instances of this problem can be investigated using MacWilliams identities for the rank support, as we now show.

Let us fix a convenient notation. The complement of a set  $I \subseteq [k] \times [m]$  is denoted by  $I^c$ . For  $I \subseteq [k] \times [m]$  define  $\text{Mat}[I] := \{M \in \text{Mat} : M_{st} = 0 \text{ for all } (s, t) \in I^c\}$ . Clearly,  $\text{Mat}[I]$  is an  $\mathbb{F}_q$ -subspace of  $\text{Mat}$  of dimension  $|I|$ .

**Remark 50.** For any subset  $I \subseteq [k] \times [m]$  we have  $\text{Mat}[I]^\perp = \text{Mat}[I^c]$ . Therefore, by Theorem 46, the rank distributions of  $\text{Mat}[I]$  and  $\text{Mat}[I^c]$  determine each other.

For some sets  $I$  the rank distribution of  $\text{Mat}[I]$  is easy to compute. In these cases Theorem 46 gives a formula for the number of matrices in  $\text{Mat}$  of any given rank and zero entries on  $I$ .

**Corollary 51.** Let  $1 \leq k' \leq k$  and  $1 \leq m' \leq m$  be integers. For all  $0 \leq j \leq k$  the number of  $k \times m$  rank  $j$  matrices  $M$  over  $\mathbb{F}_q$  such that  $M_{st} = 0$  for all  $(s, t) \in \{1, \dots, k'\} \times \{1, \dots, m'\}$  is

$$q^{-k'm'} \sum_{i=0}^{\min\{k', m'\}} \begin{bmatrix} m' \\ i \end{bmatrix} \prod_{u=0}^{i-1} (q^{k'} - q^u) \sum_{s=0}^k (-1)^{j-s} q^{ms + \binom{j-s}{2}} \begin{bmatrix} k-s \\ k-j \end{bmatrix} \begin{bmatrix} k-i \\ s \end{bmatrix}.$$

*Proof.* Let  $I := [k'] \times [m']$ . The code  $\mathcal{C} := \text{Mat}[I]$  is the set of matrices whose entries are contained in the rectangular region described by  $I$ . As a consequence, for all  $0 \leq i \leq \min\{k', m'\}$ ,  $W_i(\mathcal{C}, \omega_{\text{rk}})$  is the number of  $k' \times m'$  matrices over  $\mathbb{F}_q$  with rank  $i$ , i.e.,

$$W_i(\mathcal{C}, \omega_{\text{rk}}) = \begin{bmatrix} m' \\ i \end{bmatrix} \prod_{u=0}^{i-1} (q^{k'} - q^u) \quad \text{for } 0 \leq i \leq \min\{k', m'\}.$$

For  $\min\{k', m'\} < i \leq k'$  we have  $W_i(\mathcal{C}, \omega_{\text{rk}}) = 0$ , and the result follows from Remark 50 and Theorem 46.  $\square$

Up to a permutation of rows and columns, the matrices of Corollary 51 have all their non-zero entries contained into a Ferrers board. Matrices with this property have been widely studied in the literature (see [13] among the others).

Again concerning matrices with prescribed zero entries, a question of Stanley asks for the number of invertible matrices over  $\mathbb{F}_q$  having zero diagonal entries (see the Introduction of [20]). The question was answered in [20], Proposition 2.2, where the authors provide a formula for the number of  $k \times m$  full-rank matrices over  $\mathbb{F}_q$  with zero diagonal entries. Notice that for diagonal entries of a rectangular matrix  $M$  we mean the entries of the form  $M_{ss}$  for  $1 \leq s \leq k$ . The following corollary extends Proposition 2.2 of [20].

**Corollary 52.** Let  $I \subseteq \{(s, t) \in [k] \times [m] : s = t\}$  be a set of diagonal entries. For all  $0 \leq j \leq k$  the number of  $k \times m$  matrices  $M$  over  $\mathbb{F}_q$  having rank  $j$  and  $M_{st} = 0$  for all  $(s, t) \in I$  is

$$q^{-|I|} \sum_{i=0}^{|I|} \begin{bmatrix} |I| \\ i \end{bmatrix} (q-1)^i \sum_{s=0}^k (-1)^{j-s} q^{ms + \binom{j-s}{2}} \begin{bmatrix} k-s \\ k-j \end{bmatrix} \begin{bmatrix} k-i \\ s \end{bmatrix}.$$

*Proof.* Define  $\mathcal{C} := \text{Mat}[I]$ . For  $|I| < i \leq k$  we have  $W_i(\mathcal{C}, \omega_{\text{rk}}) = 0$ , and for  $0 \leq i \leq |I|$  we have

$$W_i(\mathcal{C}, \omega_{\text{rk}}) = \begin{bmatrix} |I| \\ i \end{bmatrix} (q-1)^i.$$

Thus the formula follows from Remark 50 and Theorem 46.  $\square$

We conclude this section mentioning a concise method to compute the number of symmetric and skew-symmetric  $k \times k$  matrices of given rank over  $\mathbb{F}_q$ . These numbers were computed by Carliz in [5] and [6] and by MacWilliams in [23] using *ad-hoc* recursive arguments. Our technique employs the Möbius inversion formula and the regularity of the lattice of subspaces of  $\mathbb{F}_q^k$ , which we denote by  $\mathcal{L}$  in the following (see Example 36).

Recall that a  $k \times k$  matrix  $M$  is **symmetric** if  $M_{ij} = M_{ji}$  for all  $1 \leq i, j \leq k$  and **skew-symmetric** if  $M_{ii} = 0$  and  $M_{ij} = -M_{ji}$  for all  $1 \leq i, j \leq k$ . We denote by  $\text{Sym}$  and  $\text{s-Sym}$  the spaces of  $k \times k$  symmetric and skew-symmetric matrices over  $\mathbb{F}_q$ , respectively.

**Lemma 53.** Let  $S \subseteq \mathbb{F}_q^k$  be any  $s$ -dimensional subspace. Then  $\{M \in \text{Sym} : \sigma_{\text{rk}}(M) \subseteq (S)\}$  has dimension  $s(s+1)/2$  over  $\mathbb{F}_q$ .

*Proof.* Define  $V := \{x \in \mathbb{F}_q^k : x_i = 0 \text{ for } i > s\} \subseteq \mathbb{F}_q^k$ . There exists an  $\mathbb{F}_q$ -isomorphism  $g : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^k$  such that  $g(S) = V$ . Let  $G \in \text{GL}_k(\mathbb{F}_q)$  be the matrix associated to  $g$  with respect to the canonical basis  $\{e_1, \dots, e_k\}$  of  $\mathbb{F}_q^k$ . Since  $G$  is invertible,  $M \mapsto GMG^t$  is an  $\mathbb{F}_q$ -linear automorphism of  $\text{Sym}$  that preserves the rank support of matrices. As a consequence,  $\dim(\{M \in \text{Sym} : \sigma_{\text{rk}}(M) \subseteq (S)\}) = \dim(\{M \in \text{Sym} : \sigma_{\text{rk}}(M) \subseteq (V)\}) = s(s+1)/2$ , as claimed.  $\square$

We can now compute the number of symmetric  $k \times k$  matrices over  $\mathbb{F}_q$  of rank  $i$  as follows. For any subspace  $T \subseteq \mathbb{F}_q^k$  define  $f(T) := |\{M \in \text{Sym} : \sigma_{\text{rk}}(M) = T\}|$  and  $g(T) := \sum_{S \subseteq T} f(S)$ . By Lemma 53, for all  $S \subseteq \mathbb{F}_q^k$  we have  $g(S) = q^{s(s+1)}$ , where  $s := \dim(S)$ . Therefore applying the Möbius inversion formula ([28], Proposition 3.7.1) to the functions  $f$  and  $g$  we obtain, for any given  $i$ -dimensional subspace  $T \subseteq \mathbb{F}_q^k$ ,

$$f(T) = \sum_{S \subseteq T} g(S) \mu_{\mathcal{L}}(S, T) = \sum_{s=0}^k \sum_{\substack{S \subseteq T \\ \dim(S)=s}} q^{s(s+1)} \mu_{\mathcal{L}}(s, i) = \sum_{s=0}^k q^{\binom{s+1}{2}} \begin{bmatrix} i \\ s \end{bmatrix} (-1)^{i-s} q^{\binom{i-s}{2}}.$$

The expected result is now obtained summing over all the  $i$ -dimensional subspaces  $T \subseteq \mathbb{F}_q^k$ . A similar argument applies to skew-symmetric matrices. The final result is the following.

**Proposition 54.** The number of symmetric and skew-symmetric  $k \times k$  matrices over  $\mathbb{F}_q$  of rank  $i$  is, respectively,

$$\begin{bmatrix} k \\ i \end{bmatrix} \sum_{s=0}^k (-1)^{i-s} q^{\binom{s+1}{2} + \binom{i-s}{2}} \begin{bmatrix} i \\ s \end{bmatrix}, \quad \begin{bmatrix} k \\ i \end{bmatrix} \sum_{s=0}^k (-1)^{i-s} q^{\binom{s}{2} + \binom{i-s}{2}} \begin{bmatrix} i \\ s \end{bmatrix}.$$

One can also observe that the spaces of  $k \times k$  symmetric and skew-symmetric matrices over  $\mathbb{F}_q$  are orthogonal to each other. Therefore the rank distributions of symmetric and skew-symmetric matrices are related by a MacWilliams transformation. More precisely, the following hold.

**Corollary 55.** For all integers  $0 \leq j \leq k$  we have

$$W_j(\text{Sym}, \omega_{\text{rk}}) = q^{-\binom{k}{2}} \sum_{i=0}^k W_i(\text{s-Sym}, \omega_{\text{rk}}) \sum_{s=0}^k (-1)^{j-s} q^{ms + \binom{j-s}{2}} \begin{bmatrix} k-s \\ k-j \end{bmatrix} \begin{bmatrix} k-i \\ s \end{bmatrix}.$$

## Acknowledgement

The author is grateful to Elisa Gorla for help in improving Section 5 and the presentation of this work.

## References

- [1] G. E. Andrews, *The Theory of Partitions*. Encyclopedia of Mathematics and its Applications, vol. 2, G.C. Rota Editor. Addison-Wesley, 1976.
- [2] E. Byrne, *On the weight distribution of codes over finite rings*. Advances in Mathematics of Communications, 5 (2011), pp. 395–406.

- [3] E. Byrne, M. Greferath, M. E. O'Sullivan, *The linear programming bound for codes over finite Frobenius rings*. Designs, Codes and Cryptography, 42 (2007), pp. 289 – 301.
- [4] P. Camion, *Codes and association schemes*. In V. S. Pless and W. C. Huffman (editors), Handbook of Coding Theory, Vol. II, pp. 1441 – 1566. Elsevier (1998).
- [5] L. Carlitz, *Representations by quadratic forms in a finite field*. Duke Mathematical Journal, 21 (1954), pp. 123 – 137.
- [6] L. Carlitz, *Representations by skew forms in a finite field*. Archiv der Mathematik, 5 (1954), pp. 19 – 31.
- [7] I. Constantinescu, W. Heise, *A metric for codes over residue class rings*. Problems on Information Transmission, 33 (1997), pp. 208 – 213.
- [8] P. Delsarte, *Association schemes and  $t$ -designs in regular semilattices*. Journal of Combinatorial Theory, Series A, 2 (1976), 2, pp. 230 – 243.
- [9] P. Delsarte, *Bilinear forms over a finite field, with applications to coding theory*. Journal of Combinatorial Theory, Series A, 25 (1978), 3, pp. 226 – 241.
- [10] H. Gluesing-Luerssen, *Fourier-reflexive partitions and MacWilliams identities for additive codes*. Designs, Codes and Cryptography, 75 (2015), pp. 543 – 563.
- [11] H. Gluesing-Luerssen, *Partitions of Frobenius rings induced by the homogeneous weight*. Advances in Mathematics of Communications, 8 (2014), pp. 191 – 207.
- [12] M. Greferath, S. Schmidt, *Finite ring combinatorics and MacWilliams' Equivalence Theorem*. Journal of Combinatorial Theory, 92A (2000), pp. 17 – 28.
- [13] J. Haglund,  *$q$ -rook polynomials and matrices over finite fields*. Advances in Applied Mathematics, 20 (1998), 4, pp. 450 – 487.
- [14] A. R. Hammons, P. V. Kumar, A. R. Calderbank, N. J. A. Sloane, P. Solé, *The  $\mathbb{Z}_4$ -linearity of Kerdock, Preparata, Goethals, and related codes*. IEEE Transactions on Information Theory, 40 (1994), pp. 301 – 319.
- [15] T. Honold, I. Landjev, *MacWilliams identities for linear codes over finite Frobenius rings*. In D. Jungnickel and H. Niederreiter, editors, *Proceedings of The Fifth International Conference on Finite Fields and Applications Fq5*, Augsburg, 1999, pp. 276 – 292. Springer 2001.
- [16] W. C. Huffman, V. Pless, *Fundamentals of Error-Correcting Codes*. Cambridge University Press (2003).
- [17] A. J. Klein, J. B. Lewis, A. H. Morales, *Counting matrices over finite fields with support on skew Young diagrams and complements of Rothe diagrams*. Journal of Algebraic Combinatorics, 39 (2014), 2, pp. 429 – 456.
- [18] T. Y. Lam, *Lectures on Modules and Rings*. Graduate Text in Mathematics, vol. 189. Springer 1999.
- [19] C. Lee, *Some properties of nonbinary error-correcting codes*. IRE Transactions on Information Theory.

- [20] J. B. Lewis, R. Liu, G. Panova, A. H. Morales, S. V Sam, Y. X. Zhang, *Matrices with restricted entries and  $q$ -analogues of permutations*. Journal of Combinatorics 2 (2012), 3, pp. 355 – 396.
- [21] J. H. van Lint, *Introduction to Coding Theory*, third edition. Springer (1999).
- [22] F. J. MacWilliams, *A Theorem on the Distribution of Weights in a Systematic Code*. Bell System Technical Journal, 42 (1963), 1, pp. 79 – 94.
- [23] F. J. MacWilliams, *Orthogonal matrices over finite fields*. American Mathematical Monthly, 76 (1969), pp. 152 – 164.
- [24] F. J. MacWilliams, N. J. A. Sloane, *The Theory of Error-Correcting Codes*. North Holland Mathematical Library.
- [25] A. Ravagnani, *Rank-metric codes*. Designs, Codes and Cryptography (to appear).
- [26] D. Silva, F. R. Kschishang, *On metrics for error correction in network coding*. IEEE Transactions on Information Theory, 55 (2009), 12, pp. 5479 – 5490.
- [27] E. Spiegel, C. J. O'Donnell, *Incidence algebras*. CRC Press (1997).
- [28] P. Stanley, *Enumerative Combinatorics*, vol. 1, second ed., Cambridge Stud. Adv. Math., vol. 49, Cambridge University Press, Cambridge (2012).
- [29] J.R. Stembridge, *Counting points on varieties over finite fields related to a conjecture of Kontsevich*. Annals of Combinatorics, 2 (1998), 4, pp. 365 – 385.
- [30] V. A. Zinoviev, T. Ericson, *Fourier invariant pairs of partitions of finite abelian groups and association schemes*. Problems of Information Transmission, 45 (2009), pp. 221 – 231.